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Generalized Kac–Moody Lie algebras, free Lie algebras and the structure of the Monster Lie algebra

Elizabeth Jurisich*

Department of Mathematics, Rutgers University New Brunswick, NJ 08903, USA

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Abstract

It is shown that any generalized Kac–Moody Lie algebra $\mathfrak{g}$ that has no mutually orthogonal imaginary simple roots can be written as $\mathfrak{g} = \mathfrak{u}^+ \oplus (\mathfrak{g}_J + \mathfrak{h}) \oplus \mathfrak{u}^-$, where $\mathfrak{g}_J$ is a Kac–Moody algebra defined from a symmetrizable Cartan matrix, and $\mathfrak{u}^+$ and $\mathfrak{u}^-$ are subalgebras isomorphic to free Lie algebras over certain $\mathfrak{g}_J$-modules. The denominator identity for such an algebra $\mathfrak{g}$ is obtained by using a generalization of Witt's formula that computes the graded dimension of the free Lie algebra $\mathfrak{u}^-$ and the denominator identity known for the Kac–Moody subalgebra $\mathfrak{g}_J$. The main result and consequent proof of the denominator identity give a new proof that the radical of a generalized Kac–Moody algebra of the above type is zero. The main result is applied to the Monster Lie algebra $\mathfrak{m}$ to obtain an elegant decomposition $\mathfrak{m} = \mathfrak{u}^+ \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^-$.

Also included is a detailed discussion of Borcherds' construction of the Monster Lie algebra from a vertex algebra and an elementary proof of Borcherds' theorem relating Lie algebras with "an almost positive definite bilinear form" to generalized Kac–Moody algebras. © 1998 Elsevier Science B.V.

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1. Introduction

Generalized Kac–Moody algebras, called Borcherds algebras in [14], were investigated by Borcherds in [2]. We show that any generalized Kac–Moody algebra $\mathfrak{g}$ that has no mutually orthogonal imaginary simple roots can be written as $\mathfrak{g} = \mathfrak{u}^+ \oplus (\mathfrak{g}_J + \mathfrak{h}) \oplus \mathfrak{u}^-$, where $\mathfrak{u}^+$ and $\mathfrak{u}^-$ are subalgebras isomorphic to free Lie algebras with given generators, and $\mathfrak{g}_J$ is a Kac–Moody algebra defined from a symmetrizable Cartan matrix (see

* Present address: Department of Mathematics, 5734 University Avenue, Chicago, IL 60637, USA; E-mail address: jurisich@math.uchicago.edu.

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Theorem 5.1). There is a formula due to Witt that computes the graded dimension of a free Lie algebra where all of the generators have been assigned degree one. It is known that Witt's formula can be extended to other gradings (e.g., [5]). We present a further generalization of the formula appearing in [5]. The denominator identity for $\mathfrak{g}$ is obtained by using this generalization of Witt's formula and the denominator identity known for the Kac–Moody algebra $\mathfrak{g}_J$. In this work, we are taking $\mathfrak{g}$ to be the algebra defined by the appropriate generators and relations, rather than the quotient of this algebra by its radical. In particular, our main result and consequent proof of the denominator identity give a new proof that the radical of a generalized Kac–Moody algebra of the above type is zero. (We use the fact that the radical of $\mathfrak{g}_J$ is zero, which is Serre's theorem in the case that $\mathfrak{g}_J$ is finite-dimensional; this is the main case for us.)

The most important application of our work is to the Monster Lie algebra $\mathfrak{m}$, defined by Borcherds [4]. In fact, we show that $\mathfrak{m} = \mathfrak{u}^+ \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^-$, with $\mathfrak{u}^\pm$ free Lie algebras. This result is obtained by applying the above results to a generalized Kac–Moody algebra $\mathfrak{g}(M)$ defined from a particular matrix M, given by the inner products of the simple roots of $\mathfrak{m}$. Theorem 5.1 applied to this Lie algebra establishes that the subalgebras $\mathfrak{n}^\pm \subset \mathfrak{g}(M)$ are each the semidirect product of a one-dimensional Lie algebra and a free Lie algebra on countably many generators. The Lie algebra $\mathfrak{g}(M)$ is shown to be a central extension of the Monster Lie algebra $\mathfrak{m}$ (Theorem 6.1) constructed by Borcherds in [4]. By Theorem 6.1, the subalgebras $\mathfrak{m}^\pm$ in $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{h} \oplus \mathfrak{m}^-$ are isomorphic to the subalgebras $\mathfrak{n}^\pm \subset \mathfrak{g}(M)$. In this way we show $\mathfrak{m}$ contains two large subalgebras $\mathfrak{u}^\pm$ which are isomorphic to free Lie algebras, and $\mathfrak{m} = \mathfrak{u}^+ \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^-$. The denominator identity for $\mathfrak{m}$ (see [4]) is obtained in this paper in the manner described above for more general $\mathfrak{g}$. In this case $\mathfrak{g}_J = \mathfrak{sl}_2$, and our results give a new proof that the central extension $\mathfrak{g}(M)$ of $\mathfrak{m}$ has zero radical.

The Monster Lie algebra $\mathfrak{m}$ is of great interest because Borcherds defines and uses this Lie algebra, along with its denominator identity, to solve the following problem [4]: It was conjectured by Conway and Norton [6] that there should be an infinite-dimensional representation of the Monster simple group such that the McKay–Thompson series of the elements of the Monster group (that is, the graded traces of the elements of the Monster group as they act on the module) are equal to some known modular functions given in [6]. After the "moonshine module" $V^\natural$ for the Monster simple group was constructed [10] and many of its properties, including the determination of some of the McKay–Thompson series, were proven in [11], the nontrivial problem of computing the rest of the McKay–Thompson series of Monster group elements acting on $V^\natural$ remained. Borcherds has shown in [4] that the McKay–Thompson series are the expected modular functions.

In this paper, in preparation for our main result, we include a detailed treatment of some of Borcherds' work on generalized Kac–Moody algebras, and of that part of [4] which shows that the Monster Lie algebra has the properties that we need to prove Theorem 6.1. We now explain this exposition.

Some results such as character formulas and a denominator identity known for Kac–Moody algebras are stated in [2] for generalized Kac–Moody algebras. We found it necessary to do some extra work in order to understand fully the precise definitions and also the reasoning which are implicit in Borcherds' work on this subject. Kac in [18] gives an outline (without detail) of how to rigorously develop the theory of generalized Kac–Moody algebras by indicating that one should follow the arguments presented there for Kac–Moody algebras (see also [16]). Included in [17] is a detailed exposition of the theory of generalized Kac–Moody algebras, along the lines of [20], where the homology results of [13] (not covered in [18]) are extended to these new Lie algebras. That this can be done is mentioned and used in [4]. The homology result gives another proof of the character and denominator formulas [17].

We find it appropriate to work with the extended Lie algebra as in [13, 20] (that is, the Lie algebra with suitable degree derivations adjoined). Alternatively, one can generalize the theorems in [18]. In either of these approaches, the Cartan subalgebra is sufficiently enlarged to make the simple roots linearly independent and have multiplicity one, just as in the case of Kac–Moody algebras. Without working in the extended Lie algebra it does not seem possible to prove the denominator and character formulas for all generalized Kac–Moody algebras. This is because the matrix from which we define the Lie algebra can have linearly dependent columns (as in the case of $\widehat{\mathfrak{sl}_2}$); we may even have infinitely many columns equal. Naturally, when it makes sense to do so, we may specialize formulas obtained involving the root lattice. In this way we obtain Borcherds' denominator identity for $\mathfrak{m}$, and show its relation to our generalization of Witt's formula.

The crucial link between the Monster Lie algebra and a generalized Kac–Moody algebra defined from a matrix is provided by Theorem 4.1, which is a theorem given by Borcherds in [3]. Versions of this theorem also appear in [2, 4]. Since this theorem can be stated most neatly in terms of a canonical central extension of a generalized Kac–Moody algebra (as in [3]) we include a section on this central extension. Theorem 4.1 roughly says that a Lie algebra with an "almost positive definite bilinear form", like the Monster Lie algebra, is the homomorphic image of a canonical central extension of a generalized Kac–Moody algebra. The way that Theorem 4.1 is stated here and in [3] (as opposed to [4] where condition 4 is not used) allows us to conclude that the Monster Lie algebra has a central extension which is a generalized Kac–Moody algebra defined from a matrix. We include in this paper a completely elementary proof of Theorem 4.1. This proof is simpler than the argument in [2] and the proof indicated [18], which require the construction of a Casimir operator. Here Eq. (11), which follows immediately from the hypotheses of the theorem, is used in place of the Casimir operator.

The Monster Lie algebra is defined (see [4]) from the vertex algebra which is the tensor product of $V^\natural$ and a vertex algebra obtained from a rank two hyperbolic lattice. This construction is reviewed in Section 6.2. The infinite-dimensional representation $V^\natural$ of the Monster simple group constructed in [10] can be given the structure of a vertex operator algebra, as stated in [1] and proved in [11]. The theory of vertex algebras and vertex operator algebras is used in proving properties of the Monster Lie algebra, so the definition of vertex algebra and a short discussion of the properties of vertex algebras are given in this paper. The "no-ghost" theorem of string theory is used here, as it is in [4], to obtain an isomorphism between homogeneous subspaces of the Monster Lie algebra and the homogeneous subspaces of $V^\natural$. A reformulation of the proof of the no-ghost theorem as given in [4, 25] is presented in the appendix of this paper.

This paper is related to the work of Kang [19], where a root multiplicity formula for generalized Kac–Moody algebras is proven by using Lie algebra homology. We recover Kang's result for the class of Lie algebras studied in this paper. Other related works include that of Harada et al. [16], who present an exposition of generalized Kac–Moody algebras along the lines of [18] (their proof of Theorem 4.1, is the proof in [2] done in complete detail; see above). The recent work of the physicists Gebert and Teschner [14] explores the module theory for some basic examples of generalized Kac–Moody algebras.

2. Generalized Kac–Moody algebras

2.1. Construction of the algebra associated to a matrix

In [2] Borcherds defines the generalized Kac–Moody algebra (GKM) associated to a matrix. Statements given here without proof have been shown in detail in [17]. In addition to [2], the reader may also want to refer to [18] where an outline is given for extending the arguments given there for Kac–Moody algebras. The Lie algebra denoted $\mathfrak{g}'(A)$ in [18], which is defined from an arbitrary matrix A, is equal to the generalized Kac–Moody algebra $\mathfrak{g}(A)$, defined below, when the matrix A satisfies conditions **(C1)**–**(C3)** given below.

Remark. In [4] any Lie algebra satisfying conditions 1–3 of Theorem 4.1 is defined to be a generalized Kac–Moody algebra. In this paper the term "generalized Kac–Moody algebra" will always mean a Lie algebra defined from a matrix as in [18]. The theory presented here, based on symmetric rather than symmetrizable matrices, can be easily adapted to the case where the matrix is symmetrizable. We use symmetric matrices in order to be consistent with the work of Borcherds and because the symmetric case is sufficient for the main applications.

We will begin by constructing a generalized Kac–Moody algebra associated to a matrix.

All vector spaces are assumed to be over $\mathbb{R}$. Let I be a set, at most countable, identified with $\mathbb{Z}_+ = \{1, 2, \ldots\}$ or with $\{1, 2, \ldots, k\}$. Let $A = (a_{ij})_{i,j \in I}$ be a matrix with entries in $\mathbb{R}$, satisfying the following conditions:

(C1) A is symmetric,

(C2) if $i \neq j$ then $a_{ij} \leq 0$,

(C3) if $a_{ii} > 0$ then $\frac{2a_{ij}}{a_{ii}} \in \mathbb{Z}$ for all $j \in I$.

Let $\mathfrak{g}_0(A) = \mathfrak{g}_0$ be the Lie algebra with generators h_i, e_i, f_i, where $i \in I$, and the following defining relations:

(R1) $[h_i, h_j] = 0$,

(R2) $[h_i, e_k] - a_{ik} e_k = 0$,

(R3) $[h_i, f_k] + a_{ik} f_k = 0$,

(R4) $[e_i, f_j] - \delta_{ij} h_i = 0$

for all $i, j, k \in I$. Let $\mathfrak{h} = \sum_{i \in I} \mathbb{R} h_i$. Let $\mathfrak{n}_0^+$ be the subalgebra generated by the $\{e_i\}_{i \in I}$ and let $\mathfrak{n}_0^-$ be the subalgebra generated by the $\{f_i\}_{i \in I}$. The following proposition is proven by the usual methods for Kac–Moody algebras (see [17] or [18]).

Proposition 2.1. *The Lie algebra $\mathfrak{g}_0$ has triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$. The abelian subalgebra $\mathfrak{h}$ has a basis consisting of $\{h_i\}_{i \in I}$, and $\mathfrak{n}_0^{\pm}$ is the free Lie algebra generated by the e_i (resp. the f_i) $i \in I$. In particular, $\{e_i, f_i, h_i\}_{i \in I}$ is a linearly independent set in $\mathfrak{g}_0$.*

For all $i \neq j$ and $a_{ii} > 0$ define

$$d_{ij}^+ = (\operatorname{ad} e_i)^{1 - 2a_{ij}/a_{ii}} e_j \quad \in \mathfrak{g}_0^+, \tag{1}$$

$$d_{ij}^- = (\operatorname{ad} f_i)^{1 - 2a_{ij}/a_{ii}} f_j \quad \in \mathfrak{g}_0^-. \tag{2}$$

Let $\mathfrak{k}_0^{\pm} \subset \mathfrak{n}_0^{\pm}$ be the ideal of $\mathfrak{n}_0^{\pm}$ generated by the elements:

$$d_{ij}^{\pm}, \tag{3}$$

$$[e_i, e_j] \quad \text{if } a_{ij} = 0 \text{ (in the case of } \mathfrak{k}_0^+), \tag{4}$$

$$[f_i, f_j] \quad \text{if } a_{ij} = 0 \text{ (in the case of } \mathfrak{k}_0^-). \tag{5}$$

Note that if $a_{ii} > 0$, then the elements (4) and (5) are of type (1) and (2). The subalgebra $\mathfrak{k}_0 = \mathfrak{k}_0^+ \oplus \mathfrak{k}_0^-$ is an ideal of $\mathfrak{g}_0$ (for details adapt the proof of Proposition 3.1 in the next section).

Definition 1. The generalized Kac–Moody algebra $\mathfrak{g}(A) = \mathfrak{g}$ associated to the matrix A is the quotient of $\mathfrak{g}_0$ by the ideal $\mathfrak{k}_0 = \mathfrak{k}_0^+ \oplus \mathfrak{k}_0^-$.

Remark. In [17, 18], a generalized Kac–Moody algebra is constructed as a quotient of $\mathfrak{g}_0$ by its radical (that is, the largest graded ideal having trivial intersection with $\mathfrak{h}$). Although this fact is not used in this paper, the ideal $\mathfrak{k}_0$ is equal to the radical of $\mathfrak{g}_0$. Of course, proving that the radical of the Lie algebra $\mathfrak{g}(A)$ is zero not trivial. It is shown in [17] that the radical of $\mathfrak{g}(A)$ is zero using results from [12, 18] and a proposition proven in [2].

Let $\mathfrak{n}^+ = \mathfrak{n}_0^+/\mathfrak{k}_0^+$ and $\mathfrak{n}^- = \mathfrak{n}_0^-/\mathfrak{k}_0^-$. Proposition 2.1 implies that the generalized Kac–Moody algebra has triangular decomposition $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. The Lie algebra $\mathfrak{g}$ is given by the corresponding generators and relations. The Lie algebra $\mathfrak{g}(A)$ is a Kac–Moody algebra when the matrix A is a generalized Cartan matrix.

Let $\deg e_i = -\deg f_i = (0,\dots,0,1,0,\dots)$ where 1 appears in the ith position, and let $\deg h_i = (0,\dots)$. This induces a Lie algebra grading by $\mathbb{Z}^I$ on $\mathfrak{g}$. Degree derivations D_i are defined by letting D_i act on the degree $(n_1, n_2, \dots)$ subspace of $\mathfrak{g}$ as multiplication by the scalar n_i. Let $\mathfrak{d}$ be the space spanned by the D_i. We extend the Lie algebra $\mathfrak{g}$ by taking the semidirect product with $\mathfrak{d}$, so $\mathfrak{g}^e = \mathfrak{d} \ltimes \mathfrak{g}$. Then $\mathfrak{h}^e = \mathfrak{d} \oplus \mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}^e$, which acts via scalar multiplication on each space $\mathfrak{g}(n_1, n_2, \dots)$.

Let $\alpha_i \in (\mathfrak{h}^e)^*$ for $i \in I$ be defined by the conditions

$$[h, e_i] = \alpha_i(h) e_i \quad \text{for all } h \in \mathfrak{h}^e.$$

Note that $\alpha_j(h_i) = a_{ij}$ for all $i, j \in I$. Because we have adjoined $\mathfrak{d}$ to $\mathfrak{h}$, the α_i are linearly independent.

For all $\varphi \in (\mathfrak{h}^e)^*$ define

$$\mathfrak{g}^\varphi = \{x \in \mathfrak{g} \mid [h, x] = \varphi(h)x \ \forall h \in \mathfrak{h}^e\}.$$

If $\varphi, \psi \in (\mathfrak{h}^e)^*$ then $[\mathfrak{g}^\varphi, g^\psi] \subset \mathfrak{g}^{\varphi+\psi}$. By definition $e_i \in \mathfrak{g}^{\alpha_i}$, and $f_i \in \mathfrak{g}^{-\alpha_i}$ for all $i \in I$. If all $n_i \le 0$, or all $n_i \ge 0$ (only finitely many nonzero), it can be shown by using the same methods as for Kac–Moody algebras that

$$\mathfrak{g}^{n_1\alpha_1 + n_2\alpha_2 + \cdots} = \mathfrak{g}(n_1, n_2, \dots)$$

and $\mathfrak{g}^0 = \mathfrak{h}$. Therefore,

$$\mathfrak{g} = \coprod_{\substack{(n_1, n_2, \dots) \\ n_i \in \mathbb{Z}}} \mathfrak{g}^{n_1\alpha_1 + n_2\alpha_2 + \cdots}. \tag{6}$$

Definition 2. The *roots* of $\mathfrak{g}$ are the nonzero elements φ of $(\mathfrak{h}^e)^*$ such that $\mathfrak{g}^\varphi \neq 0$. The elements α_i are *simple roots*, and $\mathfrak{g}^\varphi$ is the *root space* of $\varphi \in (\mathfrak{h}^e)^*$.

Denote by Δ the set of roots, Δ_+ the set of *positive* roots, i.e., the non-negative integral linear combinations of α_i. Let $\Delta_- = -\Delta_+$ be the set of *negative* roots. All of the roots are either positive or negative.

The algebra $\mathfrak{g}$ has an automorphism η of order 2 which acts as -1 on $\mathfrak{h}$ and interchanges the elements e_i and f_i. By an inductive argument, as in [18] or [23], we can construct a symmetric invariant bilinear form on $\mathfrak{g}$ such that g^φ and $\mathfrak{g}^{-\varphi}$ where $\varphi \in \Delta_+$ are nondegenerately paired; however, the restriction of this form to $\mathfrak{h}$ can be degenerate. There is a character formula for standard modules of $\mathfrak{g}$ and a denominator identity (see [2, 17, 18])

$$\prod_{\varphi \in \Delta_+} (1 - \mathrm{e}^\varphi)^{\dim g^\varphi} = \sum_{w \in W} (\det w) \sum_{\gamma \in \Omega(0)} (-1)^{l(\gamma)} \mathrm{e}^{w(\rho+\gamma)-\rho}, \tag{7}$$

where $\Omega(0) \subset \Delta_+ \cup \{0\}$ is the set of all $\gamma \in \Delta_+ \cup \{0\}$ such that γ is the sum (of length zero or greater) of mutually orthogonal imaginary simple roots.

Remark. The denominator formula (7) can be specialized to the unextended Lie algebra as long as the resulting specialization is well defined.

3. A canonical central extension

It is useful to consider a certain central extension of the generalized Kac–Moody algebra. Working with the central extension defined here (which is the same as in [3]) will simplify the statement and facilitate the proof of Theorem 4.1 below. Given a matrix A satisfying **(C1)**–**(C3)** let $\hat{\mathfrak{g}}$ be the Lie algebra with generators e_i, f_i, h_{ij} for $i, j, k, l \in I$ and relations:

(R1′) $[h_{ij}, h_{kl}] = 0$,
(R2′) $[h_{ij}, e_k] - \delta_{i,j} a_{ik} e_k = 0$ and $[h_{ij}, f_k] + \delta_{i,j} a_{ik} f_k = 0$,
(R3′) $[e_i, f_j] - h_{ij} = 0$,
(R4′) $d_{ij}^{\pm} = 0$ for all $i \neq j, a_{ii} > 0$,
(R5′) If $a_{ij} = 0$ then $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$.

The elements $d_{ij}^{\pm}$ are defined by (1) and (2).

We will study this Lie algebra by first considering the Lie algebra $\hat{\mathfrak{g}}_0$ with generators h_{ij}, e_i, f_i, where $i, j \in I$, and the defining relations **(R1′)**–**(R3′)**.

Let $\hat{\mathfrak{h}} = \sum_{i,j \in I} \mathbb{R} h_{ij}$ and let $\hat{\mathfrak{n}}_0^+$ be the subalgebra generated by the e_i $i \in I$, $\hat{\mathfrak{n}}_0^-$ be the subalgebra generated by the f_i $i \in I$. We shall prove a version of Proposition 2.1.

Lemma 3.1. *The elements h_{ij} are zero unless the ith and jth columns of the matrix A are equal.*

Proof (*Also see Borcherds* [3]). The lemma follows from the Jacobi identity and relations **(R1′), (R2′)**. □

Proposition 3.1. *The Lie algebra $\hat{\mathfrak{g}}_0 = \hat{\mathfrak{n}}_0^- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_0^+$, and the abelian Lie algebra $\hat{\mathfrak{h}}$ has a basis consisting of $\{h_{ij}\}_{i,j \in I}$ such that the* ith *and* jth *columns of $A = (a_{ij})_{i,j \in I}$ are equal. The subalgebra $\hat{\mathfrak{n}}_0^{\pm}$ is the free Lie algebra generated by the e_i (resp. the f_i), $i \in I$. The set $\{e_i, f_i\}_{i,j \in I} \cup \{h_{ij}\}_{i,j \in S}$ is linearly independent where $S = \{(i,j) \in I \times I \mid a_{ki} = a_{kj}$ for all $k \in I\}$.*

Proof. As in the classical case, one constructs a sufficiently large representation of the Lie algebra. Let $\mathfrak{h}$ be the span of the elements $\{h_{ij}\}_{i,j \in I}$. Define $\alpha_j \in \mathfrak{h}^*$ as follows:

$$\alpha_j(h_{ik}) = \delta_{ik} a_{ij}.$$

Let X be the free associative algebra on the symbols $\{x_i\}_{i \in I}$. Let $\lambda \in \mathfrak{h}^*$ be such that $\lambda(h_{ij}) = 0$ if $a_{li} \neq a_{lj}$, i.e., unless the ith and jth columns of A are equal.

We define a representation of the free Lie algebra $\mathfrak{g}_F$ with generators e_i, f_i, h_{ij} on X by the following actions of the generators:

1. $h \cdot 1 = \lambda(h)$ for all $h \in \mathfrak{h}$,
2. $f_i \cdot 1 = x_i$ for all $i \in I$,
3. $e_i \cdot 1 = 0$ for all $i \in I$,
4. $h \cdot x_{i_1} \cdots x_{i_r} = (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(h) x_{i_1} \cdots x_{i_r}$ for $h \in \mathfrak{h}$,

5. $f_i \cdot x_{i_1}x_{i_2}\cdots x_{i_r} = x_i x_{i_1}x_{i_2}\cdots x_{i_r}$,
6. $e_i \cdot x_{i_1}\cdots x_{i_r} = x_{i_1}e_i \cdot x_{i_2}\cdots x_{i_r} + (\lambda - \alpha_{i_2} - \cdots - \alpha_{i_r})(h_{ii_1})x_{i_2}x_{i_3}\cdots x_{i_r}$.

Let $\hat{\mathfrak{s}}$ be the ideal generated by the elements $[h_{ij}, h_{kl}]$, $[h_{ij}, e_k] - \delta_{i,j}a_{ik}e_k$, $[h_{ij}, f_k] + \delta_{i,j}a_{ik}f_k$, and $[e_i, f_j] - h_{ij}$. Now we will show that the ideal $\hat{\mathfrak{s}}$ annihilates the module X: It is clear that $[h, h']$ is 0 on X for $h, h' \in \mathfrak{h}$. The element $[e_i, f_i] - h_{ij}$ also acts as 0 on X by the following computation:

$$
\begin{aligned}
[e_i, f_j] \cdot x_{i_1}\cdots x_{i_r} &= e_i f_j \cdot x_{i_1}\cdots x_{i_r} - f_j e_i \cdot x_{i_1}\cdots x_{i_r} \\
&= e_i \cdot x_j x_{i_1}\cdots x_{i_r} - x_j e_i \cdot x_{i_1}\cdots x_{i_r} \\
&= (\lambda - \alpha_{i_1} - \cdots - \alpha_{i_r})(h_{ij})x_{i_1}\cdots x_{i_r} \\
&= h_{ij} \cdot x_{i_1}\cdots x_{i_r}.
\end{aligned}
$$

By a similar computation $[h_{ij}, f_k] + \delta_{ij}a_{ik}f_k$ annihilates X. Now consider the action of $[h_{ij}, e_k] - \delta_{ij}a_{ik}e_k$ on X:

$$
\begin{aligned}
[h_{ij}, e_k] \cdot 1 &= h_{ij}e_k \cdot 1 - e_k h_{ij} \cdot 1 \\
&= e_k \cdot \lambda(h_{ij})1 \\
&= 0
\end{aligned}
$$

and

$$\delta_{ij}a_{ik}e_k \cdot 1 = 0.$$

Thus, $[h_{ij}, e_k] - \delta_{ij}a_{ik}e_k$ annihilates 1. Furthermore, $[h_{ij}, e_k] - \delta_{ij}a_{ik}e_k$ commutes with the action of f_l for all l, as the following computation from [3] shows

$$
\begin{aligned}
&[[h_{ij}, e_k] - \delta_{ij}a_{ik}e_k, f_l] \\
&\quad = [[h_{ij}, e_k], f_l\ ] - [\delta_{ij}a_{ik}e_k, f_l] = [h_{ij}, h_{kl}] + \delta_{ij}(a_{il} - a_{ik})h_{kl}.
\end{aligned}
$$

By the assumption on λ, h_{kl} is 0 on X unless $a_{il} = a_{ik}$, so that the above is zero. Since any $x_{i_1}\cdots x_{i_r} = f_{i_1}\cdots f_{i_r}\cdot 1$, this means that $[h_{ij}, e_k] - \delta_{ij}a_{ik}e_k$ annihilates X. Now X can be regarded as a $\hat{\mathfrak{g}}_0$-module. The remainder of the proof follows the classical argument. □

The following proposition will be used in the proof of Theorem 4.1.

Proposition 3.2. *In $\hat{\mathfrak{g}}_0$, for all $i, j, k \in I$ with $i \neq j$ and $a_{ii} > 0$*

$$[e_k, d_{ij}^-] = 0$$

and

$$[f_k, d_{ij}^+] = 0.$$

Proof. It is enough to show the first formula.

Case 1: Assume $k \neq i$ and $k \neq j$. Since h_{ki} is central if $k \neq i$

$$\begin{aligned}(\operatorname{ad} e_k)(\operatorname{ad} f_i)x &= [h_{ki}, x] + [f_i, [e_k, x]] \\ &= (\operatorname{ad} f_i)(\operatorname{ad} e_k)x,\end{aligned} \tag{8}$$

so

$$\begin{aligned}[e_k, (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} f_j] &= (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1}[e_k, f_j] \\ &= (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} h_{kj} = 0.\end{aligned}$$

The last equality holds because $k \neq j$ means h_{kj} is central.

Case 2: Assume $k = i$. By assumption $a_{ii} > 0$, thus e_i, f_i, h_{ii} generate a Lie algebra isomorphic to $\mathfrak{sl}_2$. Consider the $\mathfrak{sl}_2$-module generated by the weight vector f_j. Then if $a_{ij} = 0$ the result follows from the Jacobi identity and the fact that $[e_i, f_j] = h_{ij}$ is in the center of the Lie algebra. If $a_{ij} \neq 0$ then

$$\begin{aligned}&\operatorname{ad} e_i (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} f_j \\ &\quad = (a_{ii}/2)(2a_{ij}/a_{ii} - 1)(-2a_{ij}/a_{ii})(\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}} f_j \\ &\qquad + (-2a_{ij}/a_{ii} + 1)(\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}}(\operatorname{ad} h_{ii}) f_j \\ &\quad = 0.\end{aligned}$$

Case 3: Assume $k = j$. By (8)

$$[e_j, (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} f_j] = (\operatorname{ad} f_i)^{-2a_{ij}/a_{ii}+1} h_{jj}. \tag{9}$$

Since $[f_i, h_{jj}] = a_{ji} f_i$ it follows immediately that (9) equals zero if $a_{ij} \leq 0$. □

Let $\mathfrak{k}_0^{\pm}$ be the ideal of $\hat{\mathfrak{g}}_0{}^{\pm}$, respectively, generated by the elements which give relations (**R4′**) and (**R5′**):

$$d_{ij}^{\pm} \quad \text{for all } i \neq j,\ a_{ii} > 0,$$

$$[e_i, e_j] \quad \text{if } a_{ij} = 0 \text{ for } \mathfrak{k}_0^{+},$$

$$[f_i, f_j] \quad \text{if } a_{ij} = 0 \text{ for } \mathfrak{k}_0^{-}.$$

Proposition 3.3. *Define* $\mathfrak{k}_0 = \mathfrak{k}_0^{+} \oplus \mathfrak{k}_0^{-}$. *Then* $\mathfrak{k}_0^{\pm}$ *and* $\mathfrak{k}_0$ *are ideals of* $\hat{\mathfrak{g}}_0$.

Proof. Let $\hat{\mathfrak{g}}_0$ act on itself by the adjoint representation. To see that $\mathfrak{k}^{-}$ is an ideal, first consider the action on the generators d_{ij}^{-}. By Proposition 3.1 we have

$$\begin{aligned}\sum_{i \neq j} \mathcal{U}(\hat{\mathfrak{g}}_0) \cdot d_{ij}^{-} &= \sum_{i \neq j} \mathcal{U}(\hat{\mathfrak{g}}_0^{-}) \mathcal{U}(\hat{\mathfrak{h}}) \mathcal{U}(\hat{\mathfrak{g}}_0^{+}) \cdot d_{ij}^{-} \\ &= \sum_{i \neq j} \mathcal{U}(\hat{\mathfrak{g}}_0^{-}) \cdot d_{ij}^{-} \subset \mathfrak{k}_0^{-}.\end{aligned}$$

The equality holds by Proposition 3.2 and the fact that $h(\operatorname{ad} f_i)^N f_j = \lambda(\operatorname{ad} f_i)^N f_j$, where λ is a scalar.

We must also consider the action on the generators of the form $[f_i, f_j]$, $i \neq j$, $a_{ij} = 0$. In this case

$$\begin{aligned}[e_k[f_i, f_j]] &= [h_{ki}, f_j] + [f_i, h_{kj}]\\ &= -\delta_{ki} a_{ij} f_j + \delta_{kj} a_{ji} f_i\\ &= 0,\end{aligned}$$

i.e.,

$$e_k \cdot [f_i, f_j] = 0.$$

Then by the same argument as above

$$\sum_{i \neq j,\ a_{ij}=0} \mathscr{U}(\hat{\mathfrak{g}}_0) \cdot [f_i, f_j] \subset \mathfrak{k}_0^-.$$

By a symmetric argument, $\mathfrak{k}_0^+$ is an ideal of $\hat{\mathfrak{g}}_0$, so $\mathfrak{k}_0$ is an ideal. $\square$

The Lie algebra $\hat{\mathfrak{g}}_0/\mathfrak{k}_0$ is equal to $\hat{\mathfrak{g}}$, the Lie algebra defined above by generators and relations.

Remark. The Lie algebra $\hat{\mathfrak{g}}$ is called the universal generalized Kac–Moody algebra in [3].

Let $\mathfrak{c}$ be the ideal of $\hat{\mathfrak{g}}$ spanned by the h_{ij} where $i \neq j$; note that these elements are central. The Lie algebra $\hat{\mathfrak{g}}$ is a central extension of the Lie algebra $\mathfrak{g}$, because there is an obvious homomorphism from $\hat{\mathfrak{g}}$ to $\mathfrak{g}$ mapping generators to generators with kernel $\mathfrak{c}$. So $1 \to \mathfrak{c} \to \hat{\mathfrak{g}} \xrightarrow{-} \mathfrak{g} \to 1$ is exact. The radical of $\hat{\mathfrak{g}}$ must be zero because the radical of $\mathfrak{g}$ is zero. We have shown the following.

Theorem 3.1. *The generalized Kac–Moody algebra* $\mathfrak{g}$ *is isomorphic to* $\hat{\mathfrak{g}}/\mathfrak{c}$.

The Lie algebra $\hat{\mathfrak{g}}$ can be given a $\mathbb{Z}$-gradation defined by taking $\deg e_i = -\deg f_i = s_i \in \mathbb{Z}$ where $s_i = s_j$ if $h_{ij} \neq 0$, and $\deg h_{ij} = 0$. The automorphism η is well defined on $\hat{\mathfrak{g}}$. It follows from Propositions 2.1 and 3.1 that $\mathfrak{n}_0^{\pm} = \hat{\mathfrak{n}}_0^{\pm}$. If $\mathfrak{n}^+ = \hat{\mathfrak{g}}_0{}^+/\mathfrak{k}_0^+$ and $\mathfrak{n}^- = \hat{\mathfrak{g}}_0{}^-/\mathfrak{k}_0^-$ then we have the decomposition $\hat{\mathfrak{g}} = \mathfrak{n}^- \oplus \hat{\mathfrak{h}} \oplus \mathfrak{n}^+$.

Recall that the Lie algebra $\mathfrak{g}^e$ has an invariant bilinear form $(\cdot,\cdot)$. It is useful to define an invariant bilinear form $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$ on $\hat{\mathfrak{g}}$. For $a, b \in \hat{\mathfrak{g}}$, let $(a, b)_{\hat{\mathfrak{g}}} = (\bar{a}, \bar{b})$. Note that the span of the h_{ij} where $i \neq j$ is in the radical of the form on $\hat{\mathfrak{g}}$. The form $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$ is symmetric and invariant because the form on $\mathfrak{g}$ has these properties. Grade $\hat{\mathfrak{g}}$ by letting $\deg e_i = 1 = -\deg f_i$, and $\deg h_{ij} = 0$, so $\hat{\mathfrak{g}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathfrak{g}}_n$, where $\hat{\mathfrak{g}}_n$ is contained in $\mathfrak{n}^+$ if $n > 0$, and $\mathfrak{n}^-$ if $n < 0$. The form $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$ is nondegenerate on $\hat{\mathfrak{g}}_n \oplus \hat{\mathfrak{g}}_{-n}$, because the map given by the central extension, $x \mapsto \bar{x}$, is an isomorphism on $\mathfrak{n}^{\pm}$, and the form defined on $\mathfrak{g}^e$ is nondegenerate on $\mathfrak{g}^e_m \oplus \mathfrak{g}^e_{-m}$ for $m \in \mathbb{Z}_+$.

4. Another characterization of GKM algebras

Theorem 4.1 below is a version of Theorem 3.1 appearing in [2]. Much of the proof of the following theorem is different than the proof appearing in [2]. In particular, there is no need to define a Casimir operator in order to show that the elements $a_{ij} \leq 0$; as seen below, this follows immediately from condition 3.

Remark. In [4] Borcherds states, as a converse to the theorem, that the canonical central extensions $\hat{\mathfrak{g}}$ satisfy conditions 1–3 below, although we note that the canonical central extension of a generalized Kac–Moody algebra does not have to satisfy condition 1 for some matrices. For example, if we start with the infinite matrix whose entries are all -2, then all of the e_i must have the same degree because of condition 3. Therefore, there is no way to define a $\mathbb{Z}$-grading of $\hat{\mathfrak{g}}$ so that $\hat{\mathfrak{g}}_i$ is both finite-dimensional and satisfies condition 3.

We also note that the kernel of the map π appearing in the following theorem can be strictly larger that the span of the h_{ij}; cf. the statement of Theorem 4.1 in [4].

Theorem 4.1 (Borcherds [4]). *Let $\mathfrak{g}$ be a Lie algebra satisfying the following conditions*:

1. *$\mathfrak{g}$ can be $\mathbb{Z}$-graded as $\coprod_{i \in \mathbb{Z}} \mathfrak{g}_i$, $\mathfrak{g}_i$ is finite dimensional if $i \neq 0$, and $\mathfrak{g}$ is diagonalizable with respect to $\mathfrak{g}_0$.*
2. *$\mathfrak{g}$ has an involution ω which maps $\mathfrak{g}_i$ onto $\mathfrak{g}_{-i}$ and acts as -1 on noncentral elements of $\mathfrak{g}_0$, in particular $\mathfrak{g}_0$ is abelian.*
3. *$\mathfrak{g}$ has a Lie algebra-invariant bilinear form $(\cdot,\cdot)$, invariant under ω, such that $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are orthogonal if $i \neq -j$, and such that the form $(\cdot,\cdot)_0$, defined by $(x,y)_0 = -(x,\omega(y))$ for $x,y \in \mathfrak{g}$, is positive definite on $\mathfrak{g}_m$ if $m \neq 0$.*
4. *$\mathfrak{g}_0 \subset [\mathfrak{g},\mathfrak{g}]$.*

Then there is a central extension $\hat{\mathfrak{g}}$ of a generalized Kac–Moody algebra and a homomorphism, π, from $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$, such that the kernel of π is in the center of $\hat{\mathfrak{g}}$.

Proof. Generators of the Lie algebra $\mathfrak{g}$ are constructed, as in [2], as follows: For $m > 0$, let $\mathfrak{l}_m$ be the subalgebra of $\mathfrak{g}$ generated by the $\mathfrak{g}_n$ for $0 < n < m$, and let $\mathfrak{e}_m$ be the orthogonal complement of $\mathfrak{l}_m$ in $\mathfrak{g}_m$ under $(\cdot,\cdot)_0$. To see that $\mathfrak{e}_m$ is invariant under $\mathfrak{g}_0$ let $x \in \mathfrak{g}_0$, $y \in \mathfrak{e}_m$, and $z \in \mathfrak{l}_m$. Then $[x,z] \in \mathfrak{l}_m$, so $(y,[x,z])_0 = 0$. Since the form $(\cdot,\cdot)_0$ satisfies $([x,y],z)_0 = -(y,[\omega(x),z])_0$, i.e., is contravariant, we have $([x,y],z)_0 = 0$, which implies that $[x,y] \in \mathfrak{e}_m$. The operators induced by the action of $\mathfrak{g}_0$ on $\mathfrak{e}_m$ commute, so we can construct a basis of $\mathfrak{e}_m$ consisting of weight vectors with respect to $\mathfrak{g}_0$. The form $(\cdot,\cdot)_0$ is positive definite on $\mathfrak{e}_m$ so an orthonormal basis can be constructed. Contravariance of the form ensures that this orthonormal basis also consists of weight vectors. The union of these bases for all the $\mathfrak{e}_m$'s can be indexed by $I = \mathbb{Z}_+$, in any order, and will be denoted $\{e_i\}_{i\in I}$. Each $\mathfrak{g}_n$, $n > 0$, is in the Lie algebra generated by $\{e_i\}_{i\in I}$, as is seen by the following induction on the degree, n: For $n = 1$, $\mathfrak{g}_1 = \mathfrak{e}_1$. Now assume that the $\mathfrak{g}_n$ for all $0 < n < m$ are contained in the Lie algebra generated by the e_i, $i \in I$. The finite-dimensional space $\mathfrak{g}_m$ decomposes under $(\cdot,\cdot)_0$ as $\mathfrak{g}_m = \mathfrak{e}_m \oplus \mathfrak{e}_m^\perp$, where $\mathfrak{e}_m^\perp = \mathfrak{l}_m \cap \mathfrak{g}_m$. By the induction assumption, $\mathfrak{e}_m^\perp$ is generated by some of the e_i's, and by construction $\mathfrak{e}_m$ has a basis consisting of e_i's. Define $f_i = -\omega(e_i)$, and $h_{ij} = [e_i, f_j]$. The $\mathfrak{g}_n$ where $n < 0$ are generated by the f_i, $i\in I$.

The elements h_{ij} can be nonzero only when $\deg e_i = \deg e_j$. This is because if $\deg e_i > \deg e_j$, which can be assumed without loss of generality, then $[e_j, [e_i, f_j]] \in \mathfrak{l}_{\deg e_i}$. Thus $([e_i, f_j], [e_i, f_j])_0 = 0$ by contravariance, and $[e_i, f_j] = 0$ by the positive definiteness of $(\cdot,\cdot)_0$. Therefore, all of the h_{ij} are in $\mathfrak{g}_0$. By assumption 4, $\mathfrak{g}_0$ is generated by the h_{ij}, $i,j\in I$.

Let $k \in \operatorname{rad}(\cdot,\cdot)$ (so $k \in \mathfrak{g}_0$). Then $([k,g],[k,g])_0 = (k,[\omega(g),[k,g]])_0 = 0$ for all $g\in\mathfrak{g}_n$, $n \neq 0$. Thus, $[k,g] = 0$ by positive definiteness, and since $k\in\mathfrak{g}_0$, k must also commute with $\mathfrak{g}_0$. Therefore, the radical of the form $(\cdot,\cdot)$ is in the center of $\mathfrak{g}$.

If $i \neq j$ then $[e_i, f_j] = h_{ij}$ is contained in the center of $\mathfrak{g}$ because it is in the radical of the form $(\cdot,\cdot)$. To see this, consider $([e_i, f_j], x)$ for a homogeneous $x\in\mathfrak{g}$. We know by assumption that this is zero if $x \notin \mathfrak{g}_0$. If $x \in \mathfrak{g}_0$ then $([e_i, f_j], x) = (e_i, [f_j, x]) = c(e_i, e_j)_0 = 0$ for some real number c, as f_j is a weight vector for $\mathfrak{g}_0$, and the e_k's are orthogonal. Thus, $([e_i, f_j], x) = 0$ for all $x\in\mathfrak{g}$.

Now it must be shown that the generators constructed above satisfy the relations (**R1′**)–(**R5′**) of the central extension of a generalized Kac–Moody algebra, and the symmetric matrix with entries $(h_{ii}, h_{jj}) = a_{ij}$ satisfies conditions (**C2**) and (**C3**).

That (**R1′**) (that is, $[h_{ij}, h_{kl}] = 0$), holds is obvious because $\mathfrak{g}_0$ is abelian, and (**R3′**) is true by definition.

The relations (**R2′**) are proven by the following argument (cf. [2]): By construction, e_i is a weight vector of $\mathfrak{g}_0$, thus for some real number c, $[h_{lm}, e_i] = ce_i$ and

$$\begin{aligned} c = c(e_i, e_i)_0 &= ([h_{lm}, e_i], e_i)_0 \\ &= ([h_{lm}, e_i], f_i) \\ &= (h_{lm}, [e_i, f_i]) = (h_{lm}, h_{ii}). \end{aligned}$$

Since h_{ij} for $i \neq j$ is in the radical of the form $(\cdot,\cdot)$, we have $c = \delta_{lm} a_{li}$. Applying ω shows the relation for f_i.

To show (**R5′**) and (**C2**) let $i \neq j$. We must prove $a_{ij} \leq 0$, and if $a_{ij} = 0$ then $[e_i, e_j] = 0$ and $[f_i, f_j] = 0$. Consider the element $[e_i, e_j]\in\mathfrak{g}$, we have

$$\begin{aligned} ([e_i, e_j], [e_i, e_j])_0 &= -(e_i, [f_j, [e_i, e_j]])_0 \\ &= -a_{ij}(e_i, e_i)_0. \end{aligned} \tag{10}$$

The last equality follows from

$$\begin{aligned}[f_j,[e_i,e_j]] &= -[e_j,[f_j,e_i]] - [e_i,[e_j,f_j]] \\ &= a_{ij}e_i.\end{aligned}$$

By Eq. (10) and the positive definiteness of the form $(\cdot,\cdot)_0$ on $\mathfrak{g}_m$, $m \neq 0$, we have $a_{ij} \leq 0$, and $a_{ij} = 0$ if and only if $[e_i,e_j] = 0$. By applying ω we also show $[f_i,f_j] = 0$ in this case.

For $a_{ii} > 0$ the Lie algebra generated by e_i, f_i, h_{ii} is isomorphic to $\mathfrak{sl}_2$, rescaling e_i and h_{ii} by $2/a_{ii}$. For each $j \in I$ the element f_j generates an $\mathfrak{sl}_2$-weight module, the weights must all be integers so $[h_{ii},f_j] = (-2a_{ij}/a_{ii})f_j$ implies that $2a_{ij}/a_{ii} \in \mathbb{Z}$, thus **(C3)** is satisfied. Proposition 3.2, which holds for $\mathfrak{g}$, shows that $[e_k,d_{ij}^-] = 0$ where $d_{ij}^- = (\operatorname{ad} f_i)^{n+1} f_j$ and $n = -2a_{ij}/a_{ii}$ (note that n is positive). Contravariance of the form gives us

$$((\operatorname{ad} f_i)^{n+1} f_j, (\operatorname{ad} f_i)^{n+1} f_j)_0 = ((\operatorname{ad} f_i)^n f_j, [e_i, (\operatorname{ad} f_i)^{n+1} f_j])_0 = 0,$$

so that $(\operatorname{ad} f_i)^{1-2a_{ij}/a_{ii}} f_j = 0$ by the positive definiteness of $(\cdot,\cdot)_0$. Applying ω to d_{ij}^- gives the relation for $(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}} e_j$. This shows the relations **(R4′)**.

Denote by $\hat{\mathfrak{g}}$ the canonical central extension, defined in Section 3, of the generalized Kac–Moody algebra associated to the matrix $a_{ij} = (h_{ii},h_{jj})$. Define a homomorphism $\pi : \hat{\mathfrak{g}} \to \mathfrak{g}$, taking the generators e_i, f_i, h_{ij} in $\hat{\mathfrak{g}}$ to the generators e_i, f_i, h_{ij} in $\mathfrak{g}$. Since the generators of $\mathfrak{g}$ have been shown to satisfy the relations of $\hat{\mathfrak{g}}$ the map π is a homomorphism from $\hat{\mathfrak{g}}$ onto $\mathfrak{g}$. The bilinear form, $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$, on $\hat{\mathfrak{g}}$ satisfies

$$(e_i,f_j)_{\hat{\mathfrak{g}}} = \delta_{ij} = (e_i,f_j),$$

$$(h_{ii},h_{jj})_{\hat{\mathfrak{g}}} = a_{ij} = (h_{ii},h_{jj}).$$

The h_{ij} with $i \neq j$ are in the radical of $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$. Thus, $(x,y)_{\hat{\mathfrak{g}}} = (\pi(x),\pi(y))$ for $x,y \in \hat{\mathfrak{g}}$, because $(x,y)_{\hat{\mathfrak{g}}}$ can be reduced using invariance and the Jacobi identity to some polynomial in the (e_i,f_j) and (h_{ii},h_{jj}).

Now we determine the kernel of the map π. Let $a \in \hat{\mathfrak{g}}$ such that $a \neq 0$ and $\pi(a) = 0$ in $\mathfrak{g}$. Recall the grading $\hat{\mathfrak{g}} = \bigoplus_{n\in\mathbb{Z}} \hat{\mathfrak{g}}_n$ and the decomposition $\hat{\mathfrak{g}} = n^+ \oplus \hat{\mathfrak{h}} \oplus \mathfrak{n}^-$, thus can write $a = a_+ + a_0 + a_-$ where $a_\pm \in \mathfrak{n}^\pm, a_0 \in \hat{\mathfrak{h}}$. Thus, $\pi(a) = \pi(a_+) + \pi(a_0) + \pi(a_-) = 0$, which is still a direct sum in $\mathfrak{g}$. Therefore, $\pi(a_+) = 0, \pi(a_0) = 0, \pi(a_-) = 0$. Assume that a_+ is homogeneous and nonzero, then for some $n > 0$, $a_+ \in \hat{\mathfrak{g}}_n$ and $(a_+,x)_{\hat{\mathfrak{g}}} = (\pi(a_+),\pi(x)) = 0$ for all $x \in \hat{\mathfrak{g}}_{-n}$. Since $(\cdot,\cdot)_{\hat{\mathfrak{g}}}$ is nondegenerate on $\hat{\mathfrak{g}}_n \oplus \hat{\mathfrak{g}}_{-n}$ we have $a_+ = 0$. By a similar argument $a_- = 0$. Thus, $a = a_0 \in \hat{\mathfrak{h}}$, and $[a,h] = 0$ for all $h \in \hat{\mathfrak{h}}$. Since $\pi(a) = 0$, we have

$$(a,h_{ii})_{\hat{\mathfrak{g}}} = (\pi(a),\pi(h_{ii})) = 0 \quad \text{for all } i$$

so

$$[a,e_i] = (a,h_{ii})_{\hat{\mathfrak{g}}} e_i = 0 \quad \text{for all } i$$

similarly $[a,f_i] = 0$ for all $i \in I$. Thus, a is in the center of $\hat{\mathfrak{g}}$. □

If the radical of the form $(\cdot,\cdot)$ is zero then the elements h_{ij} are all zero and we have a homomorphism to $\mathfrak{g}$ from a generalized Kac–Moody algebra, for which character and denominator formulas have been established. By construction of the monster Lie algebra in [3], which is also discussed later in this paper, the radical of the invariant form is zero, so that the following corollary will apply to this algebra.

Corollary 4.1. *Let* $\mathfrak{g}$ *be a Lie algebra satisfying the conditions in Theorem* 4.1. *If the radical of the form on* $\mathfrak{g}$ *is zero then there is a generalized Kac–Moody algebra* $\mathfrak{l}$ *such that* $\mathfrak{l}/\mathfrak{c} = \mathfrak{g}$, *where* $\mathfrak{c}$ *is the center of* $\mathfrak{l}$.

5. Free subalgebras of GKM algebras

5.1. Free Lie algebras

Denote by $L(X)$ the free Lie algebra on a set X, if W is a vector space with basis X then we also use the notation $L(W) = L(X)$. We will assume that X is finite or countably infinite. Let $v = (v_1, v_2, \ldots)$ be an m-tuple where $m \in \mathbb{Z}_+ \cup \infty$, and $v_i \in \mathbb{N}$ satisfy $v_i = 0$ for i sufficiently large. We will use the notation $|v| = \sum_{i=1}^{m} v_i$. If T_i are indeterminates, let

$$T^v = \prod_{i\in\mathbb{Z}_+} T_i^{v_i} \in \mathbb{R}[T_1, \ldots, T_m] \quad \text{or} \quad \mathbb{R}[T_1, T_2, \ldots].$$

We will consider gradings of the Lie algebra $L(X)$ of the following type: Let $\Delta = \mathbb{Z}^m$ and assign to each element $x \in X$ a degree $\alpha \in \Delta$ i.e., specify a map $\phi : X \to \Delta$. Let n_α be the number of elements of X of degree α and assume this is finite for all $\alpha \in \Delta$. This defines a grading of $L(X)$ which we denote $L(X) = \coprod_{(\alpha\in\Delta)} L^\alpha(X)$. Let $d(\alpha) = \dim L^\alpha(X)$. We assume that $d(\alpha)$ is finite.

An example of this type of grading is the *multigradation* from Bourbaki [5], which is defined as follows: Enumerate the set X so that $X = \{x_i\}_{i\in I}$ where $I = \mathbb{Z}_+$ or $\{1, \ldots, m\}$. Denote by $\Delta_{|X|}$ the group of $|X|$-tuples. Let $\sigma : X \to \Delta_{|X|}$ be defined by $\sigma(x_i) = \varepsilon_i = (0, \ldots, 0, 1, 0, \ldots)$ where the 1 appears in position i. The map σ defines the multigradation of $L(X)$.

Given such a grading determined by the map ϕ and the group $\Delta = \mathbb{Z}^m$, if $\Delta' = \mathbb{Z}^n$ and if $\psi : \Delta \to \Delta'$ is a homomorphism, then $\psi \circ \phi : X \to \Delta'$ also defines a grading of the Lie algebra $L(X)$. It is clear that if $\alpha \in \Delta'$ then $d(\alpha) = \sum_{\beta\in\Delta, \psi(\beta)=\alpha} d(\beta)$ as $L^\alpha(X) = \coprod_{\beta\in\Delta, \psi(\beta)=\alpha} L^\beta(X)$. Notice that $d(\alpha)$ is zero unless α has all nonnegative or all nonpositive entries, if the degree of each element of X has this property. The map ψ induces a homomorphism from $\mathbb{R}[T_1, \ldots, T_m]$ to $\mathbb{R}[T_1, \ldots T_n]$, where if $\alpha \in \Delta$ then $T^\alpha \mapsto T^{\psi(\alpha)}$.

Proposition 5.1. *Let* $L(X) = \coprod_{(\alpha\in\Delta)} L^\alpha(X)$ *be a grading of the above type. Then*

$$1 - \sum_{\alpha\in\Delta} n_\alpha T^\alpha = \prod_{\alpha\in\Delta\setminus\{0\}} (1 - T^\alpha)^{d(\alpha)}. \tag{11}$$

Proof. If we consider the multigradation of $L(X)$ by $\Delta_{|X|}$, so $L(X) = \coprod_{\beta\in\Delta_{|X|}} L^\beta(X)$, the following formula is proved in [5] (for finite X, but this immediately implies the result for an arbitrary X):

$$1 - \sum_{i\in I} T_i = \prod_{\beta\in\Delta_{|X|}\setminus\{0\}} (1 - T^\beta)^{d(\beta)}. \tag{12}$$

This implies a formula for the more general type of grading given by a map $\phi : X \to \Delta$, as above. Any such map satisfies $\phi = \phi' \circ \sigma$ where $\phi' : \Delta_{|X|} \to \Delta$ is given by

$$\varepsilon_i \mapsto \phi(x_i).$$

Applying the homomorphism ϕ' to the identity (12) gives the proposition. □

Some of our results will follow from the elimination theorem in [5], which is restated here for the convenience of the reader.

Lemma 5.1 (Elimination theorem). *Let X be a set, S a subset of X and T the set of sequences $(s_1,\dots,s_n,x)$ with $n \geq 0, s_1,\dots,s_n$ in S and x in $X\setminus S$.*

(a) *The Lie algebra $L(X)$ is the direct sum as a vector space of the subalgebra $L(S)$ of $L(X)$ and the ideal $\mathfrak{a}$ of $L(X)$ generated by $X\setminus S$.*

(b) *There exists a Lie algebra isomorphism ϕ of $L(T)$ onto $\mathfrak{a}$ which maps $(s_1,\dots, s_n,x)$ to $(\operatorname{ad} s_1 \cdots \operatorname{ad} s_n)(x)$.*

As Bourbaki [5] does for two particular gradings, we obtain formulas for computing the dimension of the homogeneous subspaces of a free Lie algebra $L(X)$ graded as above by a group Δ. The formulas derived here relate the dimension of the piece of degree α with the number of generators in that degree (which is assumed finite). If $\beta\in\Delta$ can be partitioned $\beta = \sum a_\alpha\alpha$, where $a_\alpha\in\mathbb{N}$ and $\alpha\in\Delta$, then define the partitions $P(\beta,j) = \{a = (a_\alpha)_{\alpha\in\Delta} \mid \beta = \sum a_\alpha\alpha, |a| = j\}$ and $P(\beta) = \bigcup_j P(\beta,j)$. Taking log on both sides of formula (11) leads to the equations

$$-\log\left(1 - \sum_{\alpha\in\Delta} n_\alpha T^\alpha\right) = \sum_{j\geq 1}\frac{1}{j}\left(\sum_{\alpha\in\Delta} n_\alpha T^\alpha\right)^j = \sum_{j\geq 1}\frac{1}{j}\sum_{\beta\in\Delta}\sum_{a\in P(\beta,j)}\frac{|a|!}{a!}\prod_{\alpha\in\Delta} n_\alpha^{a_\alpha} T^\beta$$

and

$$-\sum_{\alpha\in\Delta} d(\alpha)\log(1 - T^\alpha) = \sum_{\alpha\in\Delta, k\geq 1}\frac{1}{k}d(\alpha)T^{k\alpha} = \sum_\beta\sum_{k|\beta}\frac{1}{k}d(\beta/k)T^\beta.$$

Thus, if $\gamma|\beta$ means $k\gamma = \beta$ then

$$\sum_{\gamma|\beta}\frac{\gamma}{\beta}d(\gamma) = \sum_{a\in P(\beta)}\frac{(|a|-1)!}{a!}n^a$$

where n^a denotes the product of the $n_\alpha^{a_\alpha}$. Applying the Möbius inversion formula gives:

Proposition 5.2. *Let Δ be a grading of the free Lie algebra $L(X)$ as in Proposition 5.1. If $d(\beta)$ is the dimension of $L^{\beta}(X)$ for $\beta \in \Delta$ then*

$$d(\beta) = \sum_{\gamma|\beta} \left(\frac{\gamma}{\beta}\right) \mu\left(\frac{\beta}{\gamma}\right) \sum_{a \in P(\gamma)} \frac{(|a|-1)!}{a!} n^{a}. \tag{13}$$

5.2. Applications to generalized Kac–Moody algebras

In this section we will show that certain generalized Kac–Moody algebras contain large subalgebras which are isomorphic to free Lie algebras. We will apply the results of the preceding section to these examples.

We will begin with an easy example. Let A be a matrix satisfying conditions (**C1**)–(**C3**) which has no $a_{ii} > 0$, and all $a_{ij} < 0$ for $i \neq j$ (this means that imaginary simple roots are not mutually orthogonal). In this case, the generalized Kac–Moody algebra $\mathfrak{g}(A)$ is equal to $\mathfrak{g}_0(A)$. By Proposition 2.1, $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ where $\mathfrak{n}^{\pm}$ are the free Lie algebras on the sets $\{e_i\}_{i\in I}$ and $\{f_i\}_{i\in I}$, respectively.

Remark. Formula (11) can be applied to the root grading of $\mathfrak{g}(A)$ to obtain the denominator identity for $\mathfrak{g}(A)$. The root multiplicities are given by (13).

If the imaginary simple roots of a generalized Kac–Moody algebra are not mutually orthogonal then $\mathfrak{n}^{\pm}$ are not in general free, but we will show they contain ideals which are isomorphic to free Lie algebras. First we will set up some notation. Let $J \subset I$ be the set $\{i \in I \mid \alpha_i \in \Delta_R\} = \{i \in I \mid a_{ii} > 0\}$. Note that the matrix $(a_{ij})_{i,j\in J}$ is a generalized Cartan matrix, let $\mathfrak{g}_J$ be the Kac–Moody algebra associated to this matrix. Then $\mathfrak{g}_J = \mathfrak{n}_J^+ \oplus \mathfrak{h}_J \oplus \mathfrak{n}_J^-$, and $\mathfrak{g}_J$ is isomorphic to the subalgebra of $\mathfrak{g}(A)$ generated by $\{e_i, f_i\}$ with $i \in J$.

Theorem 5.1. *Let A be a matrix satisfying conditions* (**C1**)–(**C3**). *Let J and $\mathfrak{g}_J$ be as above. Assume that if $i,j \in I\backslash J$ and $i \neq j$ then $a_{ij} < 0$. Then*

$$\mathfrak{g}(A) = \mathfrak{u}^+ \oplus (\mathfrak{g}_J + \mathfrak{h}) \oplus \mathfrak{u}^-,$$

where $\mathfrak{u}^- = L(\coprod_{j\in I\backslash J} \mathscr{U}(\mathfrak{n}_J^-)\cdot f_j)$ and $\mathfrak{u}^+ = L(\coprod_{j\in I\backslash J} \mathscr{U}(\mathfrak{n}_J^+)\cdot e_j)$. The $\mathscr{U}(\mathfrak{n}_J^-)\cdot f_j$ for $j \in I\backslash J$ are integrable highest weight $\mathfrak{g}_J$-modules, and the $\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j$ are integrable lowest weight $\mathfrak{g}_J$-modules.

Note that the conditions on the a_{ij} given in the theorem are equivalent to the statement that the Lie algebra has no mutually orthogonal imaginary simple roots.

Proof. We will consider $\mathfrak{n}^+$; the case of $\mathfrak{n}^-$ is shown by a similar argument or by applying the automorphism η.

By the construction in Section 2, we have

$$\mathfrak{n}^+ = L(\{e_i\}_{i\in I})/\mathfrak{k}_0^+,$$

where $\mathfrak{k}_0^+$ is generated as an ideal of $L(\{e_i\}_{i\in I})$ by the elements

$$\{(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j \mid i,j\in J, i\neq j\}$$

and

$$\{(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j \mid i\in J, j\in I\backslash J\}.$$

This is because there are no elements of type (4). Apply the elimination theorem to the free Lie algebra $L(\{e_i\}_{i\in I})$ with $S=J$. Thus,

$$L(\{e_i\}_{i\in I}) = L(\{e_i\}_{i\in J}) \ltimes \mathfrak{a},$$

where the ideal $\mathfrak{a}$ is isomorphic to the free Lie algebra on the set $X=\{\operatorname{ad} e_{i_1}\operatorname{ad} e_{i_2}\cdots \operatorname{ad} e_{i_k}e_j \mid j\in I\backslash J \text{ and } i_m\in J\}$. Let W denote the vector space with basis X, so that $\mathfrak{a}\cong L(W)$. Observe that as an $\mathfrak{h}^{\mathfrak{e}}$-module $W\cong \coprod_{j\in I\backslash J}\mathscr{U}(\mathfrak{l})e_j$, where $\mathfrak{l}$ denotes the free Lie algebra $L(\{e_i\}_{i\in J})$.

For each fixed $j\in I\backslash J$ consider the submodule

$$R_j = \coprod_{i\in J}\mathscr{U}(\mathfrak{l})(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j \subset \mathscr{U}(\mathfrak{l})e_j.$$

Thus, identifying quotient spaces with subspaces of W,

$$\begin{aligned} W &= \coprod_{j\in I\backslash J}(\mathscr{U}(\mathfrak{l})e_j/R_j \oplus R_j) \\ &= \coprod_{j\in I\backslash J}\mathscr{U}(\mathfrak{l})e_j/R_j \oplus \coprod_{j\in I\backslash J} R_j. \end{aligned}$$

Now apply the elimination theorem to the Lie algebra $L(X)=L(W)$, choosing a basis of W of the form $S_1\cup S_2$ where S_1 is a basis of the vector space $\coprod_{j\in I\backslash J}\mathscr{U}(\mathfrak{l})e_j/R_j$ and S_2 is a basis of $\coprod_{j\in I\backslash J}R_j$. Obtaining

$$L(W) = L\left(\coprod_{j\in I\backslash J}\mathscr{U}(\mathfrak{l})e_j/R_j\right)\ltimes \mathfrak{b},$$

where $\mathfrak{b}$ is the ideal of $L(W)$ that is generated by S_2, i.e., by $\coprod_{j\in I\backslash J}R_j$. So

$$L\left(\{e_i\}_{i\in I}) = L(\{e_i\}_{i\in J})\ltimes L\coprod_{j\in I\backslash J}\mathscr{U}(\mathfrak{l})e_j/R_j\right)\ltimes \mathfrak{b}.$$

Let $\mathfrak{k}_J^+$ be the ideal of $\mathfrak{l}$ generated by

$$\{(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j \mid i,j\in J, i\neq j\},$$

then $\mathscr{U}(\mathfrak{n}_J^+)=\mathscr{U}(\mathfrak{l}/\mathfrak{k}_J^+)=\mathscr{U}(\mathfrak{l})/\mathscr{K}$, where $\mathscr{K}$ denotes the ideal of $\mathscr{U}(\mathfrak{l})$ generated by $\mathfrak{k}_J^+\subset\mathscr{U}(\mathfrak{l})$. Thus, we can decompose the vector space

$$\mathscr{U}(\mathfrak{l})e_j/R_j = \mathscr{U}(\mathfrak{n}_J^+)e_j\Big/\coprod_{i\in J}\mathscr{U}(\mathfrak{n}_J^+)(\operatorname{ad} e_i)^{1-2a_{ij}/a_{ii}}e_j\oplus\mathscr{K}e_j.$$

Applying the elimination theorem once again, using the above decomposition we obtain

$$L\left(\coprod_{j\in I\setminus J}\mathscr{U}(\mathfrak{l})e_j/R_j\right)=L\left(\coprod_{j\in I\setminus J}\left(\mathscr{U}(\mathfrak{n}_J^+)e_j\Big/\coprod_{i\in J}\mathscr{U}(\mathfrak{n}_J^+)(\operatorname{ad}e_i)^{1-2a_{ij}/a_{ii}}e_j\right)\right)\rtimes\mathfrak{c},$$

where $\mathfrak{c}$ is the ideal in $L(\coprod_{j\in I\setminus J}\mathscr{U}(\mathfrak{l})e_j/R_j)$ generated by the sum of the $\mathscr{K}e_j$. Each $\mathfrak{h}^e$-module $\mathscr{U}(\mathfrak{n}_J^+)e_j/\coprod_{i\in J}\mathscr{U}(\mathfrak{n}_J^+)(\operatorname{ad}e_i)^{1-2a_{ij}/a_{ii}}e_j$ is an integrable lowest weight module for the Lie algebra $\mathfrak{g}_J$, denoted by $\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j$, with lowest weight α_j. Thus, we have a decomposition into semidirect products:

$$L(\{e_i\}_{i\in I})=L(\{e_i\}_{i\in J})\rtimes\left[\left(L\left(\coprod_{i\in I\setminus J}\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j\right)\rtimes\mathfrak{c}\right)\rtimes\mathfrak{b}\right].$$

It is clear that, as ideals of $L(\{e_i\}_{i\in I})$, $\mathfrak{b},\mathfrak{c}\subset\mathfrak{k}_0^+$, and $\mathfrak{k}_J^+$ is $\mathfrak{k}_0^+\cap L(\{e_i\}_{i\in J})$. Therefore, since all elements of $\mathfrak{k}_0^+$ are zero in $L(\coprod_{i\in I\setminus J}\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j)$,

$$L(\{e_i\}_{i\in I})/\mathfrak{k}_0^+=L(\{e_i\}_{i\in J})/\mathfrak{k}_J^+\rtimes L\left(\coprod_{j\in I\setminus J}\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j\right).$$

By the definition of $\mathfrak{g}_J$, $\mathfrak{n}_J^+=L(\{e_i\}_{i\in J})/\mathfrak{k}_J^+$. □

Corollary 5.1. *Let A be a matrix satisfying conditions* **(C1)–(C3)**. *Assume that the matrix A has only one positive diagonal entry, $a_{ii}>0$, and if $a_{mj}=0$ then $m=i$, or $j=i$ or $m=j$. Let $S=\{(\operatorname{ad}e_i)^l e_j\}_{0\le l\le -2a_{ij}/a_{ii}}$. The subalgebra $\mathfrak{n}^+\subset\mathfrak{g}(A)$ is the semidirect product of a one-dimensional Lie algebra and a free Lie algebra, $\mathfrak{n}^+=\mathbb{R}e_i\oplus L(S)$. Similarly, $\mathfrak{n}^-=\mathbb{R}f_i\oplus L(\eta(S))$. Thus,*

$$\mathfrak{g}(A)=L(S)\oplus(\mathfrak{sl}_2+\mathfrak{h})\oplus L(\eta(S)).$$

The root grading is a grading of the type considered in Proposition 5.1 because we have the correspondence $\alpha_i\mapsto(0,\dots,1,\dots,0)$, where 1 appears in the ith place. This is the grading (6). The denominator formula given in the next result is the same as (7), after the change of variables $e^\alpha\mapsto T^\alpha$.

Corollary 5.2. *Let A be as in Theorem* 5.1, *n_α and T^α as in Proposition* 5.1. *Let $\Delta_+^J=\Delta\cap\sum_{i\in J}\mathbb{Z}_+\alpha_i$. The Lie algebra $\mathfrak{g}(A)$ has denominator formula given by* (11) *and the denominator formula for $\mathfrak{g}_J$:*

$$\begin{aligned}&\prod_{\varphi\in\Delta_+}(1-T^\varphi)^{\dim\mathfrak{g}^\varphi}\\&=\prod_{\varphi\in\Delta_+^J}(1-T^\varphi)^{\dim\mathfrak{g}_J^\varphi}\cdot\prod_{\varphi\in\Delta_+\setminus\Delta_+^J}(1-T^\varphi)^{\dim L^\varphi\left(\coprod_{j\in I\setminus J}\mathscr{U}(\mathfrak{n}_J^+)\cdot e_j\right)}\\&=\left(\sum_{w\in W}(-1)^{l(w)}T^{w\rho-\rho}\right)\left(1-\sum_{\varphi\in\Delta^+\setminus\Delta_+^J}n_\varphi T^\varphi\right).\end{aligned}$$

Obtaining the denominator formula in this way provides an alternative proof that the radical of the Lie algebra $\mathfrak{g}$ associated to the matrix A is zero. This is because Corollary 5.2 uses only the description of $\mathfrak{g}$ in terms of generators and relations, while the previous proof of the denominator formula (see [17] or [18]) is valid after the radical of the Lie algebra has been factored out. Thus, we have shown, for the particular type of matrix in Theorem 5.1:

Corollary 5.3. *Let A be as in Theorem 5.1. The generalized Kac–Moody algebra $\mathfrak{g}(A)$ has zero radical.*

Remark. A proof of the fact that any generalized Kac–Moody algebra has zero radical can be found in [16] or [17]. In both cases the argument of [12] or [18] is extended to include generalized Kac–Moody algebras by making use of a lemma appearing in [2].

Remark. If we apply (13) to $L(\coprod_{j\in I\setminus J} \mathcal{U}(\mathfrak{n}_j^+)\cdot e_j)$ we obtain Kang's [19] multiplicity formulas for the special case of generalized Kac–Moody algebras with no mutually orthogonal imaginary simple roots.

5.3. A Lie algebra related to the modular function j

We will apply our results to an important example of a generalized Kac–Moody algebra $\mathfrak{g}(M)$, defined in Section 6.2 below. It will be shown that, if $\mathfrak{c}$ denotes the center of the Lie algebra $\mathfrak{g}(M)$, then $\mathfrak{g}(M)/\mathfrak{c}$ is the Monster Lie algebra.

Recall that the modular function j has the expansion $j(q) = \sum_{i\in\mathbb{Z}} c(i)q^i$, where $c(i) = 0$ if $i < -1$, $c(-1) = 1$, $c(0) = 744$, $c(1) = 196884$. Let $J(v) = \sum_{i\geq -1} c(i)v^i$ be the formal Laurent series associated to $j(q) - 744$.

Let M be the symmetric matrix of blocks indexed by $\{-1, 1, 2, \ldots\}$, where the block in position (i,j) has entries $-(i+j)$ and size $c(i)\times c(j)$. Thus

$$M = \left(\begin{array}{c|ccc|ccc|c}
2 & 0 & \cdots & 0 & -1 & \cdots & -1 & \cdots \\ \hline
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & -2 & \cdots & -2 & -3 & \cdots & -3 & \\ \hline
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots \\
-1 & -3 & \cdots & -3 & -4 & \cdots & -4 & \\ \hline
\vdots & & \vdots & & & \vdots & &
\end{array}\right).$$

Definition 3. Let $\mathfrak{g}(M)$ be the generalized Kac–Moody algebra associated to the matrix M given above.

We have the standard decomposition

$$\mathfrak{g}(M) = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-. \tag{14}$$

The generators of $\mathfrak{g}(M)$ will be written e_{jk}, f_{jk}, h_{jk}, indexed by integers j,k where $j \in \{-1\} \cup \mathbb{Z}_+$ and $1 \le k \le c(j)$. Since there is only one e, f or h with $j = -1$, we will write these elements as e_{-1}, f_{-1}, h_{-1}. From the construction of $\mathfrak{g}(M)$ we see that the simple roots in $(\mathfrak{h}^e)^*$ are $\alpha_{-1}, \alpha_{11}, \alpha_{12}, \ldots, \alpha_{1c(1)}, \alpha_{21}, \ldots, \alpha_{2c(2)}$, etc. Note that for fixed i the functionals α_{ij} and α_{ik} agree on all of $\mathfrak{h}$ for $1 \le j,k \le c(i)$. The α_{ik} for $i > -1$ are simple imaginary roots, and the root α_{-1} is the one real simple root.

Remark. We explain the relationship between our definition of simple root and that appearing in [4]. If the restrictions of the simple roots to $\mathfrak{h}$ are denoted $\alpha_{-1}, \alpha_1, \alpha_2, \ldots$, these elements of $\mathfrak{h}^*$ correspond to the notion of "simple imaginary roots of multiplicity greater than one" in [4]. The "simple root" α_i has "multiplicity" $c(i)$ in Borcherds' terminology. Fortunately, in this case, nonsimple roots $\alpha \in (\mathfrak{h}^e)^*$ do not restrict to any α_i, also "multiplicities" do not become infinite, "roots" remain either positive or negative, etc. The functionals α_i are linearly dependent, in fact they span a two-dimensional space. The root lattice is described in [4] as the lattice $\mathbb{Z} \oplus \mathbb{Z}$ with the inner product given by

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

The "simple roots" of this Lie algebra are denoted $(1,n)$ where $n = -1$ or $n \in \mathbb{Z}_+$. However, in most cases (including the case of $\widehat{\mathfrak{sl}_2}$) serious problems arise when we do not work in a sufficiently large Cartan subalgebra. (We do not wish to write down a "denominator identity" where some of the terms are ∞.) Here, we always work in $(\mathfrak{h}^e)^*$, taking specializations when they are illuminating, as in the case of the denominator identity for $\mathfrak{g}(M)$ given below.

Corollary 5.1 applied to the Lie algebra $\mathfrak{g}(M)$ gives the following.

Theorem 5.2. *The subalgebra $\mathfrak{n}^+ \subset \mathfrak{g}(M)$ is the semidirect product of a one-dimensional Lie algebra and a free Lie algebra, so $\mathfrak{n}^+ = \mathbb{R}e_{-1} \oplus L(S)$. Similarly, $\mathfrak{n}^- = \mathbb{R}f_{-1} \oplus L(S')$. Hence,*

$$\mathfrak{g}(M) = L(S) \oplus (\mathfrak{sl}_2 + \mathfrak{h}) \oplus L(S').$$

Here $S = \bigcup_{j \in \mathbb{N}} \{(\operatorname{ad} e_{-1})^l e_{jk} \mid 0 \le l < j, 1 \le k \le c(j)\}$, and $S' = \eta(S)$.

Now that we have established that $\mathfrak{n}^+$ is the direct sum of a one dimensional space and an ideal isomorphic to a free Lie algebra we shall obtain the denominator formula for the Lie algebra $\mathfrak{g}(M)$.

Corollary 5.4. *The denominator formula for the Lie algebra* $\mathfrak{g}(M)$ *is*

$$\prod_{\varphi\in\Delta_+}(1-T^{\varphi})^{\dim\mathfrak{g}^{\varphi}} = (1-T^{\alpha_{-1}})\prod_{\varphi\in\Delta_+\setminus\{\alpha_{-1}\}}(1-T^{\varphi})^{\dim L^{\varphi}(S)}$$

$$= (1-T^{\alpha_{-1}})\left(1-\sum_{\substack{j\in\mathbb{Z}_+\\ 1\le k\le c(j),\, 0\le l<j}} T^{l\alpha_{-1}+\alpha_{jk}}\right), \tag{15}$$

which has specialization

$$u(J(u)-J(v)) = \prod_{\substack{i\in\mathbb{Z}_+\\ j\in\mathbb{Z}_+\cup\{-1\}}}(1-u^i v^j)^{c(ij)} \tag{16}$$

under the map $\phi : \Delta \to \mathbb{Z}\times\mathbb{Z}$ *determined by* $\alpha_{ik}\mapsto(1,i)$, *where we write* $T^{\alpha_{ik}}\mapsto uv^i$.

Proof of Corollary 5.4. For the denominator identity simply apply Corollary 5.2 to the root grading. The $\mathbb{Z}\times\mathbb{Z}$-grading of $L(S)$ given above is such that a generator $(\operatorname{ad} e_{-1})^l e_{j_k}$ has degree $l(1,-1)+(1,j)=(l+1,j-l)$ with $l<j$. The number of generators of degree (i,j) is $c(i+j-1)$. Applying Eq. 11 (which is the same as specializing the second product of the denominator identity via $T^{\alpha_i}\mapsto uv^i$) gives the formula

$$1-\sum_{(i,j)\in\mathbb{N}^2\setminus\{0\}} c(i+j-1)u^i v^j = \prod_{(i,j)\in\mathbb{N}^2\setminus\{0\}}(1-u^i v^j)^{\dim L^{(i,j)}}.$$

To obtain the specialization of the denominator formula of $\mathfrak{g}$ we must include the degree $(1,-1)$ subspace $\mathfrak{g}^{(1,-1)}=\mathbb{R}e_{-1}$, which is one-dimensional:

$$\prod_{(i,j)}(1-u^i v^j)^{\dim\mathfrak{g}^{(i,j)}} = \prod_{(i,j)\in\mathbb{N}^2-\{0\}}(1-u^i v^j)^{\dim L^{(i,j)}}(1-u/v)$$

$$= \left(1-\sum_{(i,j)\in\mathbb{N}^2-\{0\}} c(i+j-1)u^i v^j\right)(1-u/v)$$

$$= 1-\sum c(i+j-1)u^i v^j - u/v + \sum c(i+j-1)u^{i+1}v^{j-1}$$

$$= u(J(u)-J(v)).$$

There is a product formula for the modular function j (see [4]) which can be written:

$$p(j(p)-j(q)) = \prod_{\substack{i=1,2,\ldots\\ j=-1,1,\ldots}}(1-p^i q^j)^{c(ij)}, \tag{17}$$

which converges on an open set in $\mathbb{C}$, and so implies the corresponding identity for formal power series. Now we conclude that $\dim\mathfrak{g}^{(i,j)}=c(ij)$. □

Note that here we must know the number theory identity of [4], to determine the dimension of the root spaces of $\mathfrak{g}$.

The identity (16) is the specialization $e^{\alpha_i} \mapsto uv^i$ of the denominator identity as it appears when we apply Eq. (7) to $\mathfrak{g}(M)$.

Remark. The matrix M can be replaced by any symmetric matrix with the same first row (and column) as M with all remaining blocks having entries strictly less than zero, as long as the minor obtained by removing the first row and column has the same rank as the corresponding minor of M.

Remark. Now we apply Eq. (13) to the $\mathbb{N} \times \mathbb{N}$-grading, and the Lie algebra $L(S)$, where $S = \{(\operatorname{ad} e_{-1})^l e_{jk} \mid 0 \leq l < j, 1 \leq k \leq c(j)\}$ and there are $c(i+j-1)$ generators of degree (i,j). Since we already know the dimension of $L^{(i,j)}$ is $c(ij)$, we recover (see [19]) the following relations between the coefficients of j:

$$c(ij) = \sum_{\substack{k \in \mathbb{Z}_+ \\ k(m,n)=(i,j)}} \frac{1}{k}\mu(k) \sum_{a \in P(m,n)} \frac{(\sum a_{rs} - 1)!}{\prod a_{rs}!} \prod c(r+s-1)^{a_{rs}}. \tag{18}$$

6. The Monster Lie algebra

6.1. Vertex operator algebras and vertex algebras

For a detailed discussion of vertex operator algebras and vertex algebras the reader should consult [8, 9, 11] and the announcement [1]. Results stated here without proof can either be found in [8, 9, 11] or follow without too much difficulty from the results appearing there.

Definition 4. A *vertex operator algebra*, $(V, Y, \mathbf{1}, \omega)$, consists of a vector space V, distinguished vectors called the *vacuum vector* $\mathbf{1}$ and the *conformal vector* ω, and a linear map $Y(\cdot, z) : V \to (\operatorname{End} V)[[z, z^{-1}]]$ which is a generating function for operators v_n, i.e., for $v \in V$, $Y(v,z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1}$, satisfying the following conditions:

(V1) $V = \coprod_{n \in \mathbb{Z}} V_{(n)}$; for $v \in V_{(n)}$, $n = \operatorname{wt}(v)$,
(V2) $\dim V_{(n)} < \infty$ for $n \in \mathbb{Z}$,
(V3) $V_{(n)} = 0$ for n sufficiently small,
(V4) If $u, v \in V$ then $u_n v = 0$ for n sufficiently large,
(V5) $Y(\mathbf{1}, z) = 1$,
(V6) $Y(v,z)\mathbf{1} \in V[[z]]$ and $\lim_{z \to 0} Y(v,z)\mathbf{1} = v$, i.e., the *creation property* holds,
(V7) the following *Jacobi identity* holds:

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y(u, z_1)Y(v, z_2) - z_0^{-1}\delta\left(\frac{z_2 - z_1}{-z_0}\right) Y(v, z_2)Y(u, z_1)$$

$$= z_2^{-1}\delta\left(\frac{z_1 - z_0}{z_2}\right) Y(Y(u, z_0)v, z_2). \tag{19}$$

The following conditions relating to the vector ω also hold:

(V8) The operators ω_n generate a Virasoro algebra, i.e., if we let $L(n) = \omega_{n+1}$ for $n \in \mathbb{Z}$ then

$$[L(m), L(n)] = (m-n)L(m+n) + (1/12)(m^3 - m)\delta_{m+n,0}(\operatorname{rank} V), \tag{20}$$

(V9) If $v \in V_{(n)}$ then $L(0)v = (\operatorname{wt} v)v = nv$,

(V10) $(d/dz)Y(v,z) = Y(L(-1)v, z)$.

Definition 5. A *vertex algebra* $(V, Y, 1, \omega)$ is a vector space V with all of the above properties except for **(V2)** and **(V3)**.

Remark. This definition is a variant, with ω, of Borcherds' original definition of vertex algebra in [1].

An important class of examples of vertex algebras (and vertex operator algebras) are those associated with lattices. For the sake of the reader who may be unfamiliar with the notation we will briefly review this construction in the case of an even lattice. For complete details (and more generality) the reader may consult [11] or [8]. Given an even lattice L one can construct a vertex algebra V_L with underlying vector space:

$$V_L = S(\hat{\mathfrak{h}}_{\mathbb{Z}}^{-}) \otimes \mathbb{R}\{L\}.$$

Here we take $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{R}$, and $\hat{\mathfrak{h}}_{\mathbb{Z}}^{-}$ is the negative part of the Heisenberg algebra (with c central) defined by

$$\hat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{n \in \mathbb{Z}} \mathfrak{h} \otimes t^n \oplus \mathbb{R}c \subset \mathfrak{h} \otimes \mathbb{R}[[t]] \oplus \mathbb{R}c.$$

Therefore,

$$\hat{\mathfrak{h}}_{\mathbb{Z}}^{-} = \coprod_{n<0} \mathfrak{h} \otimes t^n.$$

The symmetric algebra on $\hat{\mathfrak{h}}_{\mathbb{Z}}^{-}$ is denoted $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^{-})$. Given a central extension of L by a group of order 2, i.e.,

$$1 \to \langle \kappa | \kappa^2 = 1 \rangle \to \hat{L} \bar{\to} L \to 1,$$

with commutator map given by $\kappa^{\langle \alpha, \beta \rangle}$, $\alpha, \beta \in L$, $\mathbb{R}$ is given the structure of a nontrivial $\langle \kappa \rangle$-module. Define $\mathbb{R}\{L\}$ to be the induced representation $\operatorname{Ind}_{\langle \kappa \rangle}^{\hat{L}} \mathbb{R}$.

If $a \in \hat{L}$ denote by $\iota(a)$ the element $a \otimes 1 \in \mathbb{R}\{L\}$. We will use the notation $\alpha(n) = \alpha \otimes t^n \in S(\hat{\mathfrak{h}}_{\mathbb{Z}}^{-})$. The vector space V_L is spanned by elements of the form

$$\alpha_1(-n_1)\alpha_2(-n_2) \dots \alpha_k(-n_k)\iota(a),$$

where $n_i \in \mathbb{N}$. The space V_L, equipped with $Y(v,z)$ as defined in [11] satisfies properties **(V1)** and **(V4)**–**(V10)**, so is a vertex algebra with conformal vector ω. Features of $Y(v,z)$ to keep in mind from [11] are: $\alpha(-1)_n = \alpha(n)$ for all $n \in \mathbb{Z}$; the $\alpha(n)$ for $n < 0$ act by left multiplication on $u \in V_L$, and

$$\alpha(n)\iota(a) = \begin{cases} 0 & \text{if } n>0, \\ \langle \alpha, \bar{a} \rangle \iota(a) & \text{if } n = 0. \end{cases}$$

Definition 7. A bilinear form $(\cdot,\cdot)$ on a vertex algebra V is *invariant* (in the sense of [9]) if it satisfies

$$(Y(v,z)w_1, w_2) = (w_1, Y(\mathrm{e}^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w_2). \tag{21}$$

(By definition $x^{L(0)}$ acts on a homogeneous element $v \in V$ as multiplication by $x^{wt(v)}$.) Such a form satisfies $(u,v) = 0$ unless $\mathrm{wt}(u) = \mathrm{wt}(v)$.

Lemma 6.1. *Let L be an even unimodular lattice. There is a nondegenerate symmetric invariant bilinear form $(\cdot,\cdot)$ on V_L.*

Proof. The vertex algebra V_L is a module for itself under the adjoint action. In fact, V_L is an irreducible module and any irreducible module of V_L is isomorphic to V_L [7].

In order to define the contragredient module note that V_L is graded by the lattice L as well as by weights, and that under this double grading $\dim V^r_{L_{(n)}} < \infty$ for $r \in L$, $n \in \mathbb{Z}$. Let $V_L' = \coprod_{n\in\mathbb{Z}, r\in L} (V^r_{L_{(n)}})^*$, the restricted dual of V_L. Denote by $\langle \cdot,\cdot \rangle$ the natural pairing between V_L and V_L'. Results of [9] pertaining to adjoint vertex operators and the contragredient module now apply to V_L'. In particular, the space V_L' can be given the structure of a V_L-module (V_L', Y') via

$$\langle Y'(v,z)w', w \rangle = \langle w', Y(e^{zL(1)}(-z^{-2})^{L(0)}v, z^{-1})w \rangle$$

for $v, w \in V_L$ $w' \in V_L'$. Since the adjoint module V_L is irreducible, the contragredient module V_L' is also irreducible. By the result of [7] quoted above, V_L is isomorphic to V_L' as a V_L-module, which is equivalent to V_L having a nondegenerate invariant bilinear form (see [9, Remark 5.3.3]). $\square$

The "moonshine module" $V^\natural$ is an infinite-dimensional representation of the Monster simple group constructed and shown to be a vertex operator algebra in [11]. The graded dimension of $V^\natural$ is $J(q)$. There is a positive definite bilinear form $(\cdot,\cdot)$ on $V^\natural$. The vertex operator algebra $V^\natural$ satisfies all of the conditions of the no-ghost theorem (Theorem 6.2), with G taken to be the Monster simple group. This vertex operator algebra will be essential to the construction of the Monster Lie algebra.

Lemma 6.2. *The positive-definite form $(\cdot,\cdot)$ on $V^\natural$ defined in* [11] *is invariant.*

Proof. There is a unique up to constant multiple nondegenerate symmetric invariant bilinear form on $V^\natural$ [22]. Fix such a form $(\cdot,\cdot)_1$ by taking $(1,1)_1 = 1$.

Let $u \in V^{\natural}_{(2)}$, a homogeneous element of weight 2. By invariance

$$(u_n w_1, w_2)_1 = (w_1, u_{-n+2} w_2)_1$$

for $w_1, w_2 \in V^{\natural}$. We claim

$$(u_n w_1, w_2) = (w_1, u_{-n+2} w_2) \tag{22}$$

$w_1, w_2 \in V^{\natural}$. In order to prove Eq. (22) we recall the construction and properties of $V^{\natural}$ of [11]. Let $x_a^+ = \iota(a) + \iota(a^{-1})$ for $a \in \Lambda$ (the Leech lattice),

$$\mathfrak{k} = S^2(\mathfrak{h} \otimes t^{-1}) \oplus \sum_{a \in \hat{\Lambda}_4} \mathbb{R} x_a^+,$$

and let $\mathfrak{p}$ be the space of elements of V_{λ}^T (the "twisted space") of weight 2. Then $V^{\natural}_{(2)} = \mathfrak{k} \oplus \mathfrak{p}$. The action $Y(v,z)$ of elements $v \in \mathfrak{p}$ is determined by conjugating by certain elements of $\mathbb{M}$ the Monster simple group (see 12.3.8 and 12.3.9 of [11]). Conjugation by these elements map $v \in \mathfrak{p}$ to $\mathfrak{k}$. Since the form $(\cdot,\cdot)$ is invariant under $\mathbb{M}$, it is sufficient to check that Eq. (22) holds for elements of $\mathfrak{k}$. Therefore, it suffices to check Eq. (22) for two types of elements $x_a^+, a \in \hat{\Lambda}_4$ and $g(-1)^2, g(-1) \in \mathfrak{h} \otimes t^{-1}$. For $u = x_a^+$ Eq. (22) follows immediately from Proposition 12.5.1 of [11]. For $u = g(-1)^2$

$$\begin{aligned} Y(u,z) &= {}^{\circ}_{\circ} g(z)^2 {}^{\circ}_{\circ} \\ &= g(z)^- g(z) + g(z) g(z)^+. \end{aligned}$$

Using $(g(i)w_1, w_2) = (w_1, g(-i)w_2)$ one computes the adjoint of $g(z)^- g(z) + g(z)g(z)^+$ which is

$$g(z^{-1})g(z^{-1})^+ + g(z^{-1})^- g(z^{-1}) = Y(u, z^{-1}) z^{-4}.$$

Thus, Eq. (22) holds for all $u \in \mathfrak{k}$, and so for all $u \in V^{\natural}_{(2)}$.

Now recall that $V^{\natural}_{(2)} = \mathscr{B}$, the Griess algebra, and the notation $\hat{\mathscr{B}}$ for the commutative affinization of the algebra $\mathscr{B}$,

$$\hat{\mathscr{B}} = \mathscr{B} \otimes \mathbb{R}[t, t^{-1}] \oplus \mathbb{R} e,$$

where t is an indeterminate and $e \neq 0$ (with nonassociative product given in [11]). By Theorem 12.3.1 [11] $V^{\natural}$ is an irreducible graded $\hat{\mathscr{B}}$-module, under

$$\begin{aligned} &\pi : \hat{\mathscr{B}} \to \operatorname{End} V^{\natural}, \\ &v \otimes t^n \mapsto x_v(n) \quad v \in \mathscr{B}, \\ &e \mapsto 1. \end{aligned}$$

Schur's lemma then implies that any nondegenerate symmetric bilinear form satisfying Eq. (22) is unique up to multiplication by a constant. Thus, we can conclude that $(\cdot,\cdot)_1 = (\cdot,\cdot)$, since the length of the vacuum is one with respect to each form. $\square$

Given V a vertex operator algebra, or a vertex algebra with ω and therefore an action of the Virasoro algebra, let

$$P_{(i)} = \{v \in V \mid L(0)v = iv, L(n)v = 0 \text{ if } n > 0\}.$$

Thus, $P_{(i)}$ consists of the lowest weight vectors for the Virasoro algebra of weight i. Then $P_{(1)}/L(-1)P_{(0)}$ is a Lie algebra with bracket given by $[u + L(-1)P_{(0)}, v+L(-1)\, P_{(0)}] = u_0 v + L(-1)P_{(0)}$. If the vertex algebra V has an invariant bilinear form $(\cdot,\cdot)$ this induces a form $(\cdot,\cdot)_{Lie}$ on the Lie algebra $P_1/L(-1)P_{(0)}$, because $L_{-1}P_{(0)} \subset \operatorname{rad}(\cdot,\cdot)$. Invariance of the form on the vertex algebra implies for $u, v \in P_{(1)}$:

$$(u_0 v, w) = -(v, u_0 w). \tag{23}$$

So that the induced form is invariant on the Lie algebra $P_{(1)}/L(-1)P_{(0)}$.

Tensor products of vertex operator algebras are again vertex operator algebras (see [9]), and more generally, by [8], the tensor product of vertex algebras is also a vertex algebra. Given two vertex algebras $(V, Y, \mathbf{1}_V, \omega_V)$ and $(W, Y, \mathbf{1}_W, \omega_W)$ the vacuum of $V \otimes W$ is $\mathbf{1}_V \otimes \mathbf{1}_W$ and the conformal vector ω is given by $\omega_V \otimes \mathbf{1}_W + \mathbf{1}_V \otimes \omega_W$. If the vertex algebras V and W both have invariant forms then it is not difficult to show that the form on $V \otimes W$ given by the product of the forms on V and W is also invariant in the sense of Eq. (21).

6.2. The Monster Lie algebra

We will review the construction of the Monster Lie algebra given in [4]. Then we give a theorem regarding its structure as a quotient of $\mathfrak{g}(M)$. Let $\mathscr{L} = \mathbb{Z} \oplus \mathbb{Z}$ with bilinear form $\langle \cdot,\cdot \rangle$ given by the matrix

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Remark. $\mathscr{L}$ is the rank two Lorentzian lattice, denoted in [4] as $II_{1,1}$.

Fix a symmetric invariant bilinear form $(\cdot,\cdot)$ on $V_{\mathscr{L}}$, normalized by taking $(\mathbf{1},\mathbf{1}) = -1$. The reason that we choose this normalization is that the resulting invariant bilinear form on the Monster Lie algebra will have the usual values with respect to the Chevalley generators, and so that the contravariant bilinear form defined below will be positive definite and not negative definite on nonzero weight spaces. Denote by $(\cdot,\cdot)$ the symmetric invariant bilinear form on $V^\natural \otimes V_{\mathscr{L}}$ given by the product of the invariant bilinear forms on $V^\natural$ and $V_{\mathscr{L}}$.

Definition 7. The *Monster Lie algebra* $\mathfrak{m}$ is defined by

$$\mathfrak{m} = P_1/\operatorname{rad}(\cdot,\cdot)_{Lie} = (P_1/L_{-1}P_0)/\operatorname{rad}(\cdot,\cdot)_{Lie}.$$

When no confusion will arise, we will use the same notation for the invariant form on the vertex algebra, and for the induced form on the Lie algebra.

Note that, by invariance $(e^r, e^s) = (-1)^{\langle r,r\rangle/2}\langle 1, (e^r)_{(-1+\langle r,r\rangle)} e^s\rangle = 0$ unless $r = -s \in \mathscr{L}$. Therefore, the induced form on $\mathfrak{m}$ satisfies the condition that $\mathfrak{m}_r$ be orthogonal to $\mathfrak{m}_s$ if $r \neq -s \in \mathscr{L}$.

Definition 8. Let θ be the involution of $V_{\mathscr{L}}$ given by $\theta\iota(a) = (-1)^{wt(a)}\iota(a^{-1})$ and $\theta(\alpha(n)) = -\alpha(n)$. This induces an involution θ on all of $V^\natural \otimes V_{\mathscr{L}}$ by letting $\theta(u \otimes ve^r) = u \otimes \theta(ve^r)$. Use the same notation for the involution induced by θ on $\mathfrak{m}$.

Note that $\theta : \mathfrak{m}_r \to \mathfrak{m}_{-r}$ if $r \neq 0$.

Let $(\cdot,\cdot)_0$ be the contravariant bilinear form on $V^\natural \otimes V_{\mathscr{L}}$ given by $(u, v)_0 = -(u, \theta(v))$, $u, v \in V^\natural \otimes V_{\mathscr{L}}$. We also denote by $(\cdot,\cdot)_0$ the contravariant bilinear form on $\mathfrak{m}$ given by $(u, v)_0 = -(u, \theta(v))_{Lie}$, $u, v \in \mathfrak{m}$.

Elements of $\mathfrak{m}$ can be written as $\sum u \otimes ve^r$, where $u \in V^\natural$ and $ve^r = v\iota(e^r) \in V_{\mathscr{L}}$. Here, a section of the map $\hat{\mathscr{L}} \bar{\to} \mathscr{L}$ has been chosen so that $e^r \in \hat{\mathscr{L}}$ satisfies $\overline{e^r} = r \in \mathscr{L}$. There is a grading of $\mathfrak{m}$ by the lattice defined by $\deg(u \otimes ve^r) = r$.

Recall the definition of the Lie algebra $\mathfrak{g}(M)$ and the standard decomposition.

Theorem 6.1. *Let $\mathfrak{c}$ denote the center of the Lie algebra $\mathfrak{g}(M)$. Then*

$$\mathfrak{g}(M)/\mathfrak{c} = \mathfrak{m}.$$

There is a triangular decomposition $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{h} \oplus \mathfrak{m}^-$, where $\mathfrak{h} \cong \mathbb{R} \oplus \mathbb{R}$. The subalgebras $\mathfrak{m}^\pm$ are isomorphic to $\mathfrak{n}^\pm \subset \mathfrak{g}(M)$.

Theorem 6.1 is proved after the statement of the no-ghost theorem. In [4] the no-ghost theorem from string theory is used to see that the Monster Lie algebra has homogeneous subspaces isomorphic to $V^\natural_{(1+mn)}$. A precise statement of the no-ghost theorem as it is used here is provided for the reader, and a proof is given in the appendix.

Theorem 6.2 (No-ghost theorem). *Let V be a vertex operator algebra with the following properties*:

(i) *V has a symmetric invariant nondegenerate bilinear form.*

(ii) *The central element of the Virasoro algebra acts as multiplication by* 24.

(iii) *The weight grading of V is an $\mathbb{N}$-grading of V, i.e., $V = \coprod_{n=0}^\infty V_{(n)}$, and dim $V_{(0)} = 1$.*

(iv) *V is acted on by a group G preserving the above structure*; *in particular the form on V is G-invariant.*

Let $\mathscr{P}_{(1)} = \{u \in V \otimes V_{\mathscr{L}} \mid L_0 u = u,\ L_i u = 0,\ i > 0\}$. The group G acts on $V \otimes V_{\mathscr{L}}$ via the trivial action on $V_{\mathscr{L}}$. Let $\mathscr{P}^r_{(1)}$ denote the subspace of $\mathscr{P}_{(1)}$ of degree $r \in \mathscr{L}$. Then the quotient of $\mathscr{P}^r_{(1)}$ by the nullspace of its bilinear form is isomorphic as a G-module with G-invariant bilinear form to $V_{(1-\langle r,r\rangle/2)}$ if $r \neq 0$ and to $V_{(1)} \oplus \mathbb{R}^2$ if $r = 0$.

Proof of Theorem 6.1. The no-ghost theorem applied to $V^\natural$ immediately gives $\mathfrak{m}_{(m,n)} \cong V^\natural_{(mn+1)}$ if $(m, n) \neq (0, 0)$. Thus, the elements of $\mathfrak{m}_r$ where $r \neq 0$ are spanned by elements of the form $u \otimes e^r$ where $r \in \mathscr{L}$, and $u \in V^\natural$ is an element of the appropriate weight. (We will use elements of $V^\natural \otimes V_L$ to denote their equivalence classes in $\mathfrak{m}$.)

We will show that all of the conditions of Theorem 4.1 are satisfied. By considering the weights (with respect to L_0), we see that the abelian subalgebra $\mathfrak{m}_{(0,0)}$ is spanned by elements of the form $1 \otimes \alpha(-1)\iota(1)$ where $\alpha \in \mathscr{L} \otimes_{\mathbb{Z}} \mathbb{R}$. Note that $\mathfrak{m}_{(0,0)}$ is two-dimensional.

By definition $\theta = -1$ on $\mathfrak{m}_{(0,0)}$. Thus, $(\theta(x), \theta(y)) = (x, y)$ for $x, y \in \mathfrak{m}_{(0,0)}$. It also follows from the definition, and symmetry of the form on V_L that

$$\begin{aligned}(\theta(u \otimes e^r), \theta(v \otimes e^{-r})) &= (u \otimes e^{-r}, v \otimes e^r)\\ &= (u \otimes e^r, v \otimes e^{-r})\end{aligned}$$

for $u, v \in V^\natural, r \in \mathscr{L}$.

Consider $-(x, \theta(x))$ for $x \in \mathfrak{m}_r, r \neq 0$. To see that this is strictly positive it is enough to consider elements of the form $x = u \otimes e^r$ where $u \in V^\natural$. Recalling the normalization of the form on V_L,

$$\begin{aligned}(x, \theta(x)) &= (u, u)(-1)^{\langle r, r\rangle/2}(e^r, e^{-r})\\ &= (-1)^{\langle r, r\rangle}(u, u)(\mathbf{1}, (e^r)_{(-1+\langle r, r\rangle)}e^{-r})\\ &= (u, u)(\mathbf{1}, \mathbf{1}) < 0.\end{aligned}$$

Therefore, we have the desired properties of the form $(\cdot, \cdot) = (\cdot, \cdot)_{Lie}$, and the contravariant form $(\cdot, \cdot)_0$.

Now if we grade $\mathfrak{m}$, as in [4], by $i = 2m + n \in \mathbb{Z}$ then we see that $\mathfrak{m}$ satisfies the grading condition. Furthermore, condition 3 is satisfied if we take θ to be the involution.

Let $v \in V^\natural$, so that $v \otimes e^r$ is of degree $r = (m, n)$. Then

$$(1 \otimes \alpha(-1)\iota(1))_0 v \otimes e^r = v \otimes \alpha(0)e^r = \langle \alpha, r\rangle v \otimes e^r. \tag{24}$$

Thus, $1 \otimes \alpha(-1)\iota(1)$ acts as the scalar $\langle \alpha, r\rangle$ on $\mathfrak{m}_{(m,n)}$. Thus, all elements of $\mathfrak{m}_{(0,0)}$ act as scalars on the $\mathfrak{m}_{(m,n)}$. As α ranges over $\mathbb{R} \oplus \mathbb{R}$ the action distinguishes between spaces of different degree. This establishes conditions 1, 2 and 3 of Theorem 4.1.

To see that $\mathfrak{m}_{(0,0)} \subset [\mathfrak{m}, \mathfrak{m}]$, let $u, v \in V^\natural_{(2)}$ and $a = e^{(1,1)}$, $b = e^{(1,-1)}$. We will show

$$[u \otimes \iota(a), v \otimes \iota(a^{-1})] \tag{25}$$

and

$$[\iota(b), \iota(b^{-1})] \tag{26}$$

are two linearly independent vectors in $\mathfrak{m}_{(0,0)}$. Since we know that $\mathfrak{m}_{(0,0)}$ is two-dimensional, this will give condition 4 of Theorem 4.1.

By [11, 8.5.44] we have that formula (26) is $\iota(b)_0\iota(b^{-1}) = \bar{b}(-1)\iota(1)$. The formula [11, 8.5.44] also shows $\iota(a)_i\iota(a_{-1}) = 0$ unless $i \le -3$ and

$$\iota(a)_i\iota(a^{-1}) = \begin{cases} \iota(1) & \text{if } i = -3, \\ \bar{a}(-1)\iota(1) & \text{if } i = -4. \end{cases}$$

Using the Jacobi identity (19) or its component form [11, 8.8.41] and the above, we obtain for formula (25):

$$(u \otimes \iota(a))_0(v \otimes \iota(a^{-1})) = u_3 v \otimes \bar{a}(-1)\iota(1) = c\mathbf{1} \otimes \bar{a}(-1)\iota(1).$$

Since we can pick u and v such that $c \neq 0$ these vectors are linearly independent and we are done.

By definition the radical of the bilinear form on $\mathfrak{m}$ is zero, so by Corollary 4.1, $\mathfrak{m}$ is $\mathfrak{l}/\mathfrak{c}$ for some generalized Kac–Moody algebra $\mathfrak{l}$. In fact $\mathfrak{m} = \mathfrak{g}(M)/\mathfrak{c}$: Because $\mathfrak{m}_{(0,0)}$ is the image of a maximal toral subalgebra, it must also be a maximal toral subalgebra. Define *roots* of $\mathfrak{m}$ as elements $\alpha \in (\mathfrak{m}_{(0,0)})^*$ such that $[h,x] = \alpha(h)x$ for all $x \in \mathfrak{m}$. The grading given by the lattice $\mathscr{L}$ corresponds to the root grading of $\mathfrak{m}$ because we have shown that the elements of $\mathfrak{m}_{(0,0)}$ act as scalars on the $\mathfrak{m}_{(m,n)}$, and that the spaces of different degree are distinguished. It follows from the no-ghost theorem applied to $V^\natural$ that $\mathfrak{m}_{(m,n)} \cong V^\natural_{mn+1}$. We know from [11] that $\dim V^\natural_{mn+1} = c(mn)$. Consider the roots of $\mathfrak{g}(M)$ restricted to $\mathfrak{h}$. Then the dimensions of these restricted root spaces of $\mathfrak{g}(M)$ are given by $c(mn)$. By the specialization (12) of the denominator formula for $\mathfrak{g}(M)$ the generalized Kac–Moody algebra $\mathfrak{g}(M)/\mathfrak{c}$ is isomorphic to $\mathfrak{m}$. $\square$

Since the map given by Corollary 4.1 is an isomorphism on $\mathfrak{n}^\pm$ there are immediate corollaries to Theorem 5.2 and the denominator identity for $\mathfrak{g}(M)$.

Corollary 6.1. *The Monster Lie algebra* $\mathfrak{m}$ *can be written as* $\mathfrak{m} = \mathfrak{u}^+ \oplus \mathfrak{gl}_2 \oplus \mathfrak{u}^-$, *where* $\mathfrak{u}^\pm$ *are free Lie algebras with countably many generators given by Corollary* 5.1.

Corollary 6.2. *The Monster Lie algebra has the denominator formula*

$$u(J(u) - J(v)) = \prod_{\substack{i \in \mathbb{Z}_+ \\ j \in \mathbb{Z}_+ \cup \{-1\}}} (1 - u^i v^j)^{c(ij)}. \tag{27}$$

Appendix. The proof of the no-ghost theorem

In [4] it is shown how to use the no-ghost theorem from string theory to understand some of the structure of the Monster Lie algebra. The proof of that theorem, Theorem 6.2, is reproduced here with the necessary rigor and in a more algebraic context.

The space $V \otimes V_{\mathscr{L}}$ is a vertex algebra with conformal vector. Recall from Section 6.1 that elements of the Virasoro algebra acting on $V \otimes V_{\mathscr{L}}$ satisfy the relations

$$[L_i, L_j] = (i-j)L_{i+j} + \tfrac{26}{12}(i^3 - i)\delta_{i+j,0}.$$

Given a nonzero $r \in \mathscr{L}$, fix nonzero $w \in \mathscr{L}$ such that $\langle w, w\rangle = 0$ and $\langle r, w\rangle \neq 0$. Define operators K_i, $i \in \mathbb{Z}$, on $V \otimes V_{\mathscr{L}}$ by $K_i = (1 \otimes w(-1))_i = 1 \otimes w(i)$.

Let $\mathscr{A}$ be the Lie algebra generated by the operators L_i, K_i with $i \in \mathbb{Z}$. These operators satisfy the relations

$$[L_i, K_j] = -jK_{i+j},$$
$$[K_i, K_j] = 0.$$

The first relation follows from the formula (in $V_{\mathscr{L}}$) $[L_m, w(n)] = -n(w(n+m))$ of [11, 8.7.13] and the fact that $[L_m \otimes 1, 1 \otimes w(n)] = 0$. The second relation holds because $[w(i), w(j)] = \langle w, w\rangle i\delta_{i+j,0}$ [11, 8.6.42] and $\langle w, w\rangle = 0$.

Let $\mathscr{W}$ be the Virasoro subalgebra of $\mathscr{A}$ generated by the $L_i, i \in \mathbb{Z}$, and let $\mathscr{Y}$ be the abelian subalgebra generated by the $K_i, i \in \mathbb{Z}$. Denote by $\mathscr{A}^+$ the subalgebra generated by the L_i, K_i with $i > 0$, let $\mathscr{A}^-$ be the subalgebra generated by the L_i, K_i with $i < 0$, and let $\mathscr{A}^0$ be the subalgebra generated by L_0, K_0. The subalgebras $\mathscr{W}^{\pm}$, $\mathscr{W}^0$ and $\mathscr{Y}^{\pm}$, $\mathscr{Y}^0$ are defined analogously.

The vertex algebra $V \otimes V_{\mathscr{L}}$ is graded by $\mathscr{L}$, because $V_{\mathscr{L}} = S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes \mathbb{R}\{\mathscr{L}\}$ has such a grading. The subspace of degree r is $V \otimes S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-) \otimes e^r$. This space will be denoted $\mathscr{H}$. The following subspaces of the $\mathscr{A}$-module $\mathscr{H}$ will be useful:

$$\mathscr{P} = \{v \in \mathscr{H} \mid \mathscr{W}^+ v = 0\},$$
$$\mathscr{T} = \{v \in \mathscr{H} \mid \mathscr{A}^+ v = 0\},$$
$$\mathscr{N} = \text{the radical of the bilinear form } (\cdot,\cdot)_0 \text{on } \mathscr{P},$$
$$\mathscr{K} = U(\mathscr{Y})\mathscr{T}.$$

Denote $V \otimes e^r$ by Ve^r.

Lemma A.1. *With respect to the bilinear form $(\cdot,\cdot)_0$ on $V \otimes V_{\mathscr{L}}$, $L_i^* = L_{-i}$ and $K_i^* = K_{-i}$ for all $i \in \mathbb{Z}$.*

Proof. Let ω be the conformal vector of $V \otimes V_{\mathscr{L}}$, so $L_i = \omega_{i+1}$ and $\theta(\omega) = \omega$. By the definition of the form $(\cdot,\cdot)_0$ and Eq. (21) L_i^* is

$$\mathrm{Res}_{z^{-i-2}} Y(e^{zL_1}(-z^{-2})^{L_0}\omega, z^{-1}).$$

Since $L_1\omega = 0$ and $\mathrm{wt}\,\omega = 2$ we have

$$Y(e^{zL_1}(-z^{-2})^{L_0}\omega, z^{-1}) = \sum_{n \in \mathbb{Z}} \omega_n z^{n-3}.$$

Thus, $L_i^* = \omega_{-i+1} = L_{-i}$.

Now consider $K_i = (1 \otimes w(-1))_i$. By Eq. (21)

$$K_i^* = \mathrm{Res}_{z^{-i-1}} \theta Y(e^{zL_1}(-z^{-2})^{L_0}(1 \otimes w(-1)), z^{-1}).$$

To calculate this, note that $\theta(1 \otimes w(-1)) = -(1 \otimes w(-1))$, that $\mathrm{wt}(1 \otimes w(-1)) = 1$ and that $L_1(1 \otimes w(-1)) = 0$, so

$$-Y(e^{zL_1}(-z^{-2})^{L_0}(1 \otimes w(-1)), z^{-1}) = \sum_{n \in \mathbb{Z}} (1 \otimes w(-1))_n z^{n-1}.$$

We conclude $K_i^* = K_{-i}$. $\square$

Lemma A.2. *The bilinear form $(\cdot,\cdot)_0$ restricted to $\mathscr{H}$ is nondegenerate.*

Proof. The form $(\cdot,\cdot)_0$ on $V \otimes V_{\mathscr{L}}$ is nondegenerate. The form also satisfies $(u,v)_0 = 0$ unless $\deg(u) = \deg(v)$ in $\mathscr{L}$. Thus, the radical of the form $(\cdot,\cdot)_0$ restricted to $\mathscr{H}$ is contained in the radical of the form on $V \otimes V_{\mathscr{L}}$. $\square$

Lemma A.3. $\mathscr{H} = U(\mathscr{A})\mathscr{T}$.

Proof. The bilinear form on $\mathscr{H}$ is nondegenerate (Lemma A.2) and distinct L_0-weight spaces of $\mathscr{H}$ are orthogonal. Thus, the finite-dimensional ith L_0-weight space $\mathscr{H}_i = U(\mathscr{A})\mathscr{T}_i \oplus (U(\mathscr{A})\mathscr{T})_i^{\perp}$. Then there is a decomposition into $\mathscr{A}$-submodules:

$$\mathscr{H} = U(\mathscr{A})\mathscr{T} \oplus (U(\mathscr{A})\mathscr{T})^{\perp}.$$

If the graded submodule $(U(\mathscr{A})\mathscr{T})^{\perp}$ is nonempty then it contains a vector annihilated by $\mathscr{A}^+$ by the following argument: The grading of $\mathscr{H}$ (by weights of L_0) is such that $\mathscr{H} = \coprod_{i \geq (1/2)\langle r,r \rangle} \mathscr{H}_i$. The actions of the generators L_i and K_i, $i > 0$, of $\mathscr{A}^+$ lower the weight of a vector in $\mathscr{H}$. If n is the smallest integer such that $(U(\mathscr{A})\mathscr{T})^{\perp} \cap \mathscr{H}_n$ is nonzero, then this subspace consists of vectors annihilated by $\mathscr{A}^+$. By definition such a vector is in $\mathscr{T}$, hence is in $U(\mathscr{A})\mathscr{T}$, a contradiction. $\square$

Lemma A.4. $\mathscr{K} = \mathscr{T} \oplus \mathrm{rad}(\cdot,\cdot)_0$.

Proof. Note that $\mathscr{Y} = \mathscr{Y}^- \oplus \mathscr{Y}^+ \oplus \mathscr{Y}^0$, so that by the Poincare–Birkhoff–Witt theorem $\mathscr{K} = U(\mathscr{Y}^-)U(\mathscr{Y}^+)U(\mathscr{Y}^0)\mathscr{T}$. By definition of $\mathscr{T}$, $U(\mathscr{Y}^+)\mathscr{T} = \mathscr{T}$ so $\mathscr{K} = U(\mathscr{Y}^-)\mathscr{T}$. Thus,

$$\mathscr{K} = \mathscr{T} \oplus \mathscr{Y}^- U(\mathscr{Y}^-)\mathscr{T}.$$

By Lemma A.1

$$\begin{aligned}(\mathscr{K}, \mathscr{Y}^- U(\mathscr{Y}^-)\mathscr{T})_0 &= (\mathscr{Y}^+ \mathscr{K}, U(\mathscr{Y}^-)\mathscr{T})_0 \\ &= 0.\end{aligned}$$

Therefore $\mathscr{Y}^- U(\mathscr{Y}^-)\mathscr{T} \subset \mathrm{rad}(\cdot,\cdot)_0$. Furthermore, $\mathscr{T} \cap \mathrm{rad}(\cdot,\cdot)_0 = 0$ because if $t \in \mathscr{T} \cap \mathrm{rad}(\cdot,\cdot)_0$ then $(t, U(\mathscr{A})\mathscr{T})_0 = (t, U(\mathscr{A}^-)\mathscr{T})_0 = 0$, and by Lemma A.2, $t = 0$. $\square$

Lemma A.5. $\mathscr{K} = Ve^r \oplus rad(\cdot,\cdot)_0$.

Proof. It is immediate from the definition that elements of $\mathscr{K}$ are lowest weight vectors of $\mathscr{Y}$. Furthermore,

$$\begin{aligned}\mathscr{H} &= U(\mathscr{A})\mathscr{T} = [U(\mathscr{W}^-)U(\mathscr{Y}^-)]\mathscr{T} \\ &= \mathscr{W}^- U(\mathscr{W}^-)U(\mathscr{Y}^-)\mathscr{T} \oplus U(\mathscr{Y}^-)\mathscr{T}.\end{aligned}$$

Since no nonzero element of $\mathscr{W}^- U(\mathscr{W}^-)U(\mathscr{Y}^-)\mathscr{T}$ is a lowest weight vector of $\mathscr{Y}$, $\mathscr{K}$ is the subspace of $\mathscr{H}$ of lowest weight vectors of the abelian Lie algebra $\mathscr{Y}$.

In order to describe the lowest weight vectors explicitly consider $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ as a polynomial algebra on the generators $x_i = w(-i)$, $z_i = r(-i)$, $i > 0$, so that $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-) = \mathbb{C}[x_i]_{i>0} \otimes \mathbb{C}[z_i]_{i>0}$. The elements $w(i)$, $i \in \mathbb{Z}$ act on $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ via multiplication, with $w(i) \cdot 1 = 0$ if $i > 0$, $[w(k), w(j)] = 0$ and $[w(k), r(j)] = k\delta_{k+j,0}\langle w, r\rangle$. Thus, the element $w(k)$ acts on $\mathbb{C}[z_i]_{i>0}$ as the differential operator $k\langle r, w\rangle \partial/\partial z_k$ for all $k > 0$.

By definition of the K_i, the lowest weight vectors of the $\mathscr{Y}$-module $\mathscr{H}$ are the lowest weight vectors of the above actions of the $w(i)$, $i \in \mathbb{Z}$, on $\mathbb{C}[x_i]_{i>0} \otimes \mathbb{C}[z_i]_{i>0}$. Since the action of the $w(i)$, $i \in \mathbb{Z}$, commute with the elements $\mathbb{C}[x_i]_{i>0}$, the lowest weight vectors are determined by the elements $q \in \mathbb{C}[z_i]_{i>0}$ satisfying $\partial/\partial z_k q = 0$ for all $k > 0$, so q is a constant. Therefore, the lowest weight vectors of the action of $\mathscr{Y}$ on $S(\hat{\mathfrak{h}}_{\mathbb{Z}}^-)$ correspond to the elements $\mathbb{C}[x_i]_{i>0}$. Thus, $\mathscr{K} = Ve^r \oplus [V \otimes (\mathbb{C}[x_i]_{i>0} \backslash \mathbb{C})e^r]$. Furthermore, $V \otimes (\mathbb{C}[w(-i)]_{i>0} \backslash \mathbb{C})e^r \subset \mathrm{rad}(\cdot,\cdot)_0$, and the form on Ve^r is nondegenerate. $\square$

Let $\mathscr{S} = \mathscr{W}^- U(\mathscr{W}^-)U(\mathscr{Y}^-)\mathscr{T} \subset \mathscr{H}$. This is called the space of "spurious vectors" in [25]. It follows from this definition (and the Poincare–Birkhoff–Witt theorem) that $\mathscr{H} = \mathscr{S} \oplus \mathscr{K}$.

Lemma A.6. *The associative algebra generated by the elements L_i for $i > 0$ is generated by elements mapping $\mathscr{S}_{(1)}$ into $\mathscr{S}$.*

Proof. This is exactly the same argument as in [25]. First we will show that L_1 and $L_2 + \frac{3}{2}L_1^2$ have this property. Any $s \in \mathscr{S}$ can be written

$$s = L_{-1}f_1 + L_{-2}f_2,$$

where $f_1, f_2 \in \mathscr{H}$ since any L_m, $m < 0$ can be written as a polynomial in L_{-1} and L_{-2}. Furthermore, $L_0 s = s$ if and only if

$$L_0L_{-1}f_1 + L_0L_{-2}f_2 = L_{-1}f_1 + L_{-2}f_2,$$

and we may assume $L_0 f_1 = 0, L_0 f_2 = -f_2$.

Thus, $s \in \mathscr{S}_{(1)}$ and $s = L_{-1}f_1 + L_{-2}f_2$ and $L_0 f_2 = -f_2$ and $L_0 f_1 = 0$. Now we compute

$$\begin{aligned}L_1 s &= L_1L_{-1}f_1 + L_1L_{-2}f_2 \\ &= L_{-1}L_1f_1 + 2L_0f_1 + 3L_{-1}f_2 + L_{-2}L_1f_2\end{aligned}$$

and this is in $\mathscr{S}$. Furthermore,

$$\begin{aligned}
&(L_2 + \tfrac{3}{2}L_1L_1)s \\
&\quad = L_2L_{-1}f_1 + \tfrac{3}{2}L_1L_1L_{-1}f_1 + L_2L_{-2}f_2 + \tfrac{3}{2}L_1L_1L_{-2}f_2 \\
&\quad = L_{-1}L_2f_1 + 3L_1f_1 + \tfrac{3}{2}L_1L_{-1}L_1f_1 + L_{-2}L_2f_2 \\
&\qquad + 4L_0f_2 + \tfrac{26}{12}6f_2 + \tfrac{3}{2}L_1L_{-2}L_1f_2 + \tfrac{3^2}{2}L_1L_{-1}f_2 \\
&\quad = L_{-1}\left(L_2 + \tfrac{3}{2}L_1L_1\right)f_1 + L_{-2}\left(L_2 + \tfrac{3}{2}L_1L_1\right)f_2 + 9L_{-1}L_1f_2.
\end{aligned}$$

The above is a spurious vector (it contains L_{-i} with $i > 0$). Note that $D = 26$ is necessary for this computation to work. Since L_1 and $L_2 + \frac{3}{2}L_1^2$ generate the algebra generated by the L_i, where $i > 0$, the lemma is proven. □

Lemma A.7. *$\mathscr{P}_{(1)}$ is the direct sum of $\mathscr{T}_{(1)}$ and $\mathscr{N}_{(1)}$.*

Proof. Let $p \in \mathscr{P}_{(1)}$. Then $p = k + s$ where $k \in \mathscr{K}_{(1)}$ and $s \in \mathscr{S}_{(1)}$; the decomposition is unique. By the preceding lemma a generator u (that is, L_1 or $L_2 + \frac{3}{2}L_1^2$) of $\mathscr{W}^+$ satisfies $0 = up = us + uk \in \mathscr{S} \oplus \mathscr{K}$. Thus, $us = uk = 0$, and we see that s is annihilated by $\mathscr{W}^+$. We conclude that $k \in \mathscr{K} \cap \mathscr{P} = \mathscr{T}$ and $s \in \mathscr{S}_{(1)} \cap \mathscr{P}$. Since $\mathscr{S}$ is orthogonal to $\mathscr{P}$, s must be an element in the radical of the form, $s \in \mathscr{N}_{(1)}$. We conclude that $\mathscr{P}_{(1)} = \mathscr{T}_{(1)} \oplus \mathscr{N}_{(1)}$. □

Theorem 6.2 now follows for $r \neq 0$ because Lemmas A.4 and A.5 imply $Ve^r \approx \mathscr{T}$ so $V_{(1-\langle r,r\rangle/2)}e^r \approx \mathscr{T}_{(1)}$, and Lemma A.7 shows that $\mathscr{T}_{(1)}$ is isomorphic to the quotient $\mathscr{P}_{(1)}/\mathscr{N}_{(1)}$. The isomorphism is naturally a G-isomorphism.

If $r = 0$, first note that

$$(V \otimes V_{\mathscr{L}})_{(1)} = V_{(1)} \oplus (V_{(0)} \otimes (V_{\mathscr{L}})_{(1)}).$$

The subspace $(V_{\mathscr{L}})_{(1)}$ is two-dimensional, spanned by vectors of the form $\alpha(-1)\iota(1)$, $\beta(-1)\iota(1)$ where α, β span $\mathscr{L}$. Furthermore, if $v \in V_{(1)}$ then $L_nv = 0$ for $n > 1$ by the condition that V be $\mathbb{N}$-graded. Since $L_1v \in V_{(0)}$, $L_1v = c\mathbf{1}$ for some constant c and since $(L_1v, \mathbf{1}) = (v, L_{-1}\mathbf{1}) = 0$ we have $c = 0$. It is easy to show that if $v \in V_{(0)} \otimes (V_{\mathscr{L}})_{(1)}$ then $L_nv = 0$ for $n > 0$. Thus the vectors in $V_{(1)} \oplus (V_{(0)} \otimes (V_{\mathscr{L}})_{(1)})$ are in $\mathscr{P}_{(1)}$, and the radical of the form restricted to this subspace is nondegenerate so Theorem 6.2 follows. □

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